Complex Analysis, Fall 2017

Problem Set 2

Due: September 19 in class

1. If $R_1$ is the radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$ and $R_2$ is the radius of convergence of $\sum_{n=0}^{\infty} b_n z^n$, then show that the radius of convergence of $\sum_{n=0}^{\infty} a_n b_n z^n$ is at least $R_1 R_2$.

2. For what value of $z$ is
   \[ \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left( \frac{z}{1+z} \right)^n \]
   convergent?

3. Determine all values of $i^i$ and $(-1)^{2i}$.

4. Assume $\{a_n\}$ is a non-increasing sequence of real numbers, and $\lim_{n \to \infty} a_n = 0$. For any $\delta > 0$, show that $\sum_{n=0}^{\infty} a_n z^n$ is uniformly convergent on $|z - 1| \geq \delta, |z| \leq 1$.

5. Describe the image of the right half plane ($\text{Re } z > 0$) under the function $z \mapsto \log z$.

6. Prove $\cos(z+w) = \cos z \cos w - \sin z \sin w$ for any two complex numbers $w$ and $z$.

7. Let $\{a_n\}$ be the Fibonacci sequence: $a_0 = a_1 = 1$, and $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$, and let $\alpha_1$ and $\alpha_2$ be the 2 roots of the polynomial $x^2 - x - 1$.

   a) Use induction to show that there are numbers $A$ and $B$ such that
      \[ a_n = A\alpha_1^n + B\alpha_2^n \]
      for every $n \geq 0$.

   b) What is the radius of convergence $R$ of the power series $\sum_{n=0}^{\infty} a_n z^n$?
(c) What does the power series converge to in \( \{ |z| < R \} \)? (It is a general fact that a power series converges to a quotient of polynomials if and only if its coefficients satisfy a recurrence relation.)