Complex Analysis, Fall 2017

Problem Set 3

Due: September 26 in class

1. Find the linear fractional transformation which maps 1, -1, 0 to 0, i, -i.

2. Show that the union of two open connected subsets of \( \mathbb{C} \) is open and connected if and only if their intersection is non-empty.

3. Let \( H = \{ z, \text{Im}(z) > 0 \} \) be the upper half plane. Assume \( M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{GL}_2(\mathbb{R}) \) is such that \( ad - bc > 0 \), and let \( f_M \) be the corresponding linear fractional transformation. Show that \( f_M \) maps \( H \) onto \( H \).

4. Assume \( (z_1, z_2, z_3, z_4) = \lambda \). What are all the values in terms of \( \lambda \) of cross ratios that we get if we consider all the 24 permutations of \( z_1, z_2, z_3, z_4 \)? Justify your answer.

5. Let \( f = u + iv : U \to \mathbb{C} \) be a function such that the partial derivatives of \( u \) and \( v \) exist and are continuous on \( U \). Assume \( f \) preserves the magnitude of angles at \( z_0 \in U \). Show that either \( f \) is holomorphic at \( z_0 \) with \( f'(z_0) \neq 0 \), or \( \bar{f} \) is holomorphic at \( z_0 \) and \( \bar{f}'(z_0) \neq 0 \).

6. Let \( D \) be the unit disk: \( D = \{ z \mid |z| < 1 \} \).

(a) Show that a linear fractional transformation of the form

\[
f(z) = e^{-i\theta} \frac{z - \alpha}{\bar{\alpha}z + 1}, \quad \alpha \in D, \ \theta \in \mathbb{R}
\]

sends \( D \) to \( D \).

(b) Conversely show that a linear transformation which sends \( D \) to \( D \) is of the above form. (Hint: Assume \( \alpha \) and \( \beta \) are such that \( f(0) = \beta \) and \( f(\alpha) = 0 \). Find \( f(\infty) \) and \( f^{-1}(\infty) \).)