1. Compute \( \int_C \frac{2z+1}{z^2+z+1} \, dz \) where \( C \) is the circle \( |z| = 2 \) positively oriented.

2. a) Give an example to show that holomorphic functions do not always map simply connected regions to simply connected regions. b) Suppose that \( U \) a simply connected region, and \( f(z) \) a nowhere vanishing holomorphic function on \( U \). Prove that there is a holomorphic function \( g \) on \( U \) such that \( e^{g(z)} = f(z) \).

3. Show that if 0 is an isolated singular point of \( f \) and \( |f(z)| \leq \frac{1}{|z|^{1/2}} \) near 0, then 0 is a removable singular point of \( f \).

4. Prove that an isolated singularity of \( f(z) \) is removable if \( \text{Re} \ f(z) \) is bounded above or below. (Hint: show that an isolated singularity of \( f(z) \) cannot be a pole of \( e^{f(z)} \).)

5. Suppose \( U \) is a region and \( f \) is holomorphic on \( U \). Let \( z_0 \in U \) and \( f'(z_0) \neq 0 \). Prove that
\[
\frac{2\pi i}{f'(z_0)} = \oint_C \frac{1}{f(z) - f(z_0)} \, dz.
\]
where \( C \) is a small circle around \( z_0 \).

6. Let \( U = \{ z : |z| > R \} \) for a fixed positive number \( R \). We say the function \( f : U \to \mathbb{C} \) has a removable singularity, pole, or essential singularity at infinity if \( f(1/z) \) has a removable, a pole, or essential singularity at 0.

   (a) Prove that an entire function has a removable singularity at infinity if and only if it is a constant.

   (b) Prove that an entire function has a pole of order \( m \) at infinity if and only if it is a polynomial of degree \( m \).

   (c) Show that \( \sin z \) and \( \cos z \) have essential singularities at infinity.