Complex Analysis, Fall 2017

Problem Set 6

Due: October 19 in class

1. Compute $\int_C \frac{2z+1}{z^2+z+1}\,dz$ where C is the circle $|z| = 2$ positively oriented.

2. a) Give an example to show that holomorphic functions do not always map simply connected regions to simply connected regions. b) Suppose that U a simply connected region, and $f(z)$ a nowhere vanishing holomorphic function on U. Prove that there is a holomorphic function g on U such that $e^{g(z)} = f(z)$.

3. Show that if 0 is an isolated singular point of f and $|f(z)| \leq \frac{1}{|z|^{1/2}}$ near 0, then 0 is a removable singular point of f.

4. Prove that an isolated singularity of $f(z)$ is removable if Re $f(z)$ is bounded above or below. (Hint: show that an isolated singularity of $f(z)$ cannot be a pole of $e^{f(z)}$.)

5. Suppose U is a region and f is holomorphic on U. Let $z_0 \in U$ and $f'(z_0) \neq 0$. Prove that

$$\frac{2\pi i}{f'(z_0)} = \int_C \frac{1}{f(z) - f(z_0)}\,dz.$$

where C is a small circle around z_0.

6. Let $U = \{z : |z| > R\}$ for a fixed positive number R. We say the function $f : U \to \mathbf{C}$ has a *removable singularity, pole, or essential singularity at infinity* if $f(1/z)$ has a removable, a pole, or essential singularity at 0.

(a) Prove that an entire function has a removable singularity at infinity if and only if it is a constant.

(b) Prove that an entire function has a pole of order m at infinity if and only if it is a polynomial of degree m.

(c) Show that $\sin z$ and $\cos z$ have essential singularities at infinity.