Complex Analysis, Fall 2017

Problem Set 8

Due: November 17 in class

1. Find the following residues.

(a) Res_{z=1} \left( \frac{(z^3-1)(z+2)}{(z^4-1)^2} \right)

(b) Res_{z=0} \left( \frac{\sin z}{z^6} \right)

(c) Res_{z=1} \left( \frac{1}{z^n-1} \right)

2. Show that for \( n \geq 1 \)

\[
\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^{n+1}} \, dx = \frac{1 \times 3 \times \cdots \times (2n-1)}{2 \times 4 \times \cdots \times (2n)} \frac{\pi}{n}.
\]

3. Compute the following integrals over \( C = \{|z| = 8\} \) positively oriented.

(a) \( \int_C \frac{1}{1-\cos z} \, dz \)

(b) \( \int_C \frac{1+z}{1-e^z} \, dz \)

4. Using residues, compute

\[
\int_{0}^{\infty} \frac{\log x}{(x^2 + 1)^2} \, dx.
\]

(Hint: look at \( f(z) = \frac{\log z}{(z^2+1)^2} \) where the branch of logarithm is given by deleting the negative imaginary axis, so the angle is \(-\pi/2 < \theta < 3\pi/2\). Use the closed path which is the boundary of the upper semicircle of radius \( R \) with a bump of radius \( r \) avoiding the origin.)
5. Evaluate the integral
\[ \int_0^\pi \frac{1}{(a + \cos \theta)^2} \, d\theta \quad (a > 1). \]

6. Suppose that \( a \) is a real number which is not an integer. Evaluating the integral
\[ \int_{C_R} \frac{\pi \cos \pi z}{(a + z)^2 \sin \pi z} \, dz \]
over the circle \( C_R \) of radius \( R = N + \frac{1}{2}, \ N \in \mathbb{Z} \), around the origin when \( N \to \infty \), show that
\[ \sum_{n=-\infty}^{\infty} \frac{1}{(a + n)^2} = \frac{\pi^2}{(\sin \pi a)^2}. \]

Conclude that
\[ \sum_{n=0}^{\infty} \frac{1}{(2n + 1)^2} = \frac{\pi^2}{8}. \]