1. (a) Show that if \( f \) has a simple pole at \( z_0 \), then \( \text{Res}_{z=z_0}(f) = \lim_{z \to z_0} (z - z_0)f(z) \) (the limit can be computed by l'Hopital's rule.)

(b) Let \( R = \{ x + iy | -1 \leq x \leq 1 - \epsilon, 0 \leq y \leq 1 \} \subset \mathbb{C} \) where \( \epsilon \) is a small positive number, and let \( \gamma \) be the boundary of \( R \). Compute

\[
\int_{\gamma} \frac{1}{z^5 - 1} \, dz.
\]

2. Determine the number of zero of the polynomial

\[ 2z^5 - 6z^2 + z + 1 \]

in the annulus \( 1 \leq |z| \leq 2 \).

3. Let \( f \) be holomorphic on the closed unit disk \( \mathbb{D} \). Assume that \( |f(z)| = 1 \) if \( |z| = 1 \), and \( f \) is not constant. Use Rouche’s theorem to show that

(a) \( f \) has a zero in \( \mathbb{D} \).

(b) The image of \( f \) contains \( \mathbb{D} \).

4. Let \( f = u + iv \). Show that \( u \) and \( v \) are both harmonic if and only if \( \frac{\partial f}{\partial \bar{z}} \) is holomorphic.

5. Prove that a harmonic function is an open map.

6. Let \( \alpha \in \mathbb{D} \), and \( \alpha = re^{i\phi} \).
(a) Show that

$$\text{Re} \left( \frac{e^{i\theta} + \alpha}{e^{i\theta} - \alpha} \right) = \frac{1 - r^2}{1 - 2r \cos(\theta - \phi) + r^2}.$$ 

(b) Show that if \(u\) is harmonic on \(D\) and continuous on \(\overline{D}\), then

$$u(\alpha) = \frac{1}{2\pi} \int_{0}^{2\pi} P_r(\theta - \phi) u(e^{i\theta}) \, d\theta,$$

where \(P_r\) is the Poisson kernel given by

$$P_r(\eta) = \frac{1 - r^2}{1 - 2r \cos \eta + r^2}.$$

7. Show that if \(P_r\) is as in Problem 7, then for \(0 \leq r < 1\) and \(\theta \in \mathbb{R}\),

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}.$$