5. Write \( f \circ g = u + iv, \) \( g = u_1 + iv_1, \) and \( f = u_2 + iv_2. \) Then
\[
\frac{\partial (f \circ g)}{\partial z} = \frac{1}{2}[(u_x + v_y) + i(v_x - u_y)].
\]
The chain rule for partial derivatives (thinking of \( f \) and \( g \) as maps from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \)) gives
\[
\begin{bmatrix}
u_x & u_y \\
v_x & v_y
\end{bmatrix} = \begin{bmatrix} u_2x & u_2y \\
v_2x & v_2y
\end{bmatrix} \begin{bmatrix} u_1x & u_1y \\
v_1x & v_1y
\end{bmatrix}
\]
So
\[
\frac{\partial (f \circ g)}{\partial z} = \frac{1}{2}[(u_2xu_1x + u_2yv_1x + v_2xu_1y + v_2yv_1y) + i(v_2xu_1x + v_2yv_1x - u_2xu_1y - u_2yv_1y)]
\]
\[
= \frac{1}{4}[(u_2x + v_2y + i(v_2x - u_2y))(u_1x + v_1y + i(v_1x - u_1y)) +
(u_2x - v_2y + i(v_2x + u_2y))(u_1x - v_1y + i(-v_1x - u_1y))]
\]
\[
= \frac{\partial f}{\partial z} \frac{\partial g}{\partial z} + \frac{\partial f}{\partial \bar{z}} \frac{\partial g}{\partial z} + \frac{\partial f}{\partial z} \frac{\partial \bar{g}}{\partial z} + \frac{\partial f}{\partial \bar{z}} \frac{\partial \bar{g}}{\partial z}
\]
\]
6. (a) Write \( f = u + iv. \) Since \( u \) is constant on \( U, \) \( u_x = u_y = 0. \) So by Cauchy-Riemann equations \( v_x = v_y = 0, \) so \( f' = 0, \) so \( f \) is a constant.
(b) Assume \( |f| = c \) on \( U. \) If \( c = 0, \) then \( f = 0 \) on \( U. \) Assume \( c \neq 0. \) Then \( f \bar{f} = c^2. \) Taking \( \frac{\partial}{\partial z} \) we get
\[
\frac{\partial f}{\partial z} \bar{f} + f \frac{\partial \bar{f}}{\partial z} = 0.
\]
But \( \frac{\partial f}{\partial z} = \frac{\partial \bar{f}}{\partial \bar{z}}. \) Since \( f \) is holomorphic, \( \frac{\partial f}{\partial z} = 0, \) so \( \frac{\partial f}{\partial \bar{z}} = 0. \) Therefore, \( \frac{\partial f}{\partial \bar{z}} \bar{f} = 0. \) Since we are assuming \( c \neq 0, \) \( f \neq 0, \) so \( \frac{\partial f}{\partial \bar{z}} = 0 \) on \( U. \) Again since \( f \) is holomorphic \( f' = \frac{\partial f}{\partial z} = 0 \) on \( U, \) so \( f \) is constant on \( U. \)
7. This follows from the proof of Lucas’ Theorem. Let \( z_1, \ldots, z_n \) be the roots of \( f \) and \( w \) a root of \( f' \), so
\[
f = a_n(z - z_1) \cdots (z - z_n).
\]
If \( f(w) = 0 \), then \( w \) is a repeated root of \( f \), and we are done. Otherwise, assume \( f(w) \neq 0 \), and write
\[
0 = \frac{f'(w)}{f(w)} = \sum_{i=1}^{n} a_n \frac{1}{w - z_i}.
\]
Let \( L \) be the line formed by the boundary of \( \Delta(f) \) which contains \( w \). Then there is half-plane formed by \( L \) which contains all the roots of \( f \). Assume the equation of that half-plane is
\[
\{ z : \Im((w - z)e^{-i\theta}) \geq 0 \}.
\]
Then equation of the line is \( \{ z : \Im((z - w)e^{-i\theta}) = 0 \} \). If all the roots of \( f \) lie on \( L \), then we are done. Otherwise, there is \( z_i \) such that \( \Im((w - z_i)e^{-i\theta}) < 0 \). Multiplying both sides of Equation 1 by \( e^{i\theta} \) we get a contradiction since the imaginary part of each term would be non-positive, and the imaginary part of at least one term is strictly negative.

8. First Assume \( B \) is a finite number. Then for every \( \epsilon > 0 \), there is \( N \) such that for \( n \geq N \), \( B - \epsilon \leq \frac{|a_n|}{a_{n+1}} \leq B + \epsilon \). Therefore for every \( k \geq N \)
\[
(B - \epsilon)^{k-N} \leq \frac{|a_N|}{a_k} \leq (B + \epsilon)^{k-N}.
\]
So
\[
\frac{|a_N|}{(B + \epsilon)^{k-N}} \leq |a_k| \leq \frac{|a_N|}{(B - \epsilon)^{k-N}}.
\]
Therefore for every \( k \geq N \)
\[
\frac{|a_N|^\frac{1}{k}}{(B + \epsilon)^{1-\frac{N}{k}}} \leq |a_k|^\frac{1}{k} \leq \frac{|a_N|^\frac{1}{k}}{(B - \epsilon)^{1-\frac{N}{k}}}.
\]
So
\[
\frac{1}{B - \epsilon} \leq \limsup_{k \to \infty} |a_k|^\frac{1}{k} \leq \frac{1}{B + \epsilon}.
\]
Letting \( \epsilon \) go to zero, we get \( \limsup_{k \to \infty} |a_k|^\frac{1}{k} = \frac{1}{B} \).

Similarly if \( B = \infty \), then for every \( M > 0 \), there is \( N \) such that for \( n \geq N \), \( \frac{a_n}{a_{n+1}} \leq M \). So for every \( k \geq N \),
\[
\frac{a_N}{a_k} \geq M^{k-N}.
\]
So for every $k \geq N$,

$$|a_k|^{\frac{1}{k}} \leq \frac{|a_N|^{\frac{1}{k}}}{M^{1-\frac{1}{k}}}.$$ 

Therefore,

$$\limsup_{k \to \infty} |a_k|^{\frac{1}{k}} \leq \frac{1}{M},$$

so $\limsup_{k \to \infty} |a_k|^{\frac{1}{k}} = 0$. 

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