Analysis I, Fall 2017

Solutions to Problem Set 2

1. We have $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \frac{1}{R_1}$ and $\limsup_{n\to\infty} |b_n|^{\frac{1}{n}} = \frac{1}{R_2}$. For every $\epsilon > 0$ there is N such that if $n \geq N$, $|a_n|^{\frac{1}{n}} < \frac{1}{R_1} + \epsilon$ and $|b_n|^{\frac{1}{n}} < \frac{1}{R_2} + \epsilon$. So for every $n \geq N$

$$|a_n b_n|^{\frac{1}{n}} \leq \frac{1}{R_1 R_2} + \epsilon(\frac{1}{R_1} + \frac{1}{R_2}) + \epsilon^2.$$

Therefore $\limsup_{n\to\infty} |a_n b_n|^{\frac{1}{n}} \leq \frac{1}{R_1 R_2}$.

2. Let $w = \frac{z}{1+z}$, and $a_n = \frac{1}{\sqrt{n}}$. Then since $\lim a_n = 0$ and a_n is non-increasing, by a convergence test that we proved in class, the series is convergence if $|w| \leq 1$ and $w \neq 1$, and obviously if $|w| > 1$ or $w = 1$, the series is divergent since the limit of the terms is not zero. But w cannot be equal to 1, so the series is convergent whenever $|\frac{z}{1+z}| \leq 1$. That is $|z|^2 \leq |1+z|^2$. Let $z = x + iy$, then we want $x^2 + y^2 \leq (x+1)^2 + y^2$. This holds when $x \geq -\frac{1}{2}$.

4. Let $f_k(z) = \sum_{n=0}^{k} a_n z^n$. To show the statement it is enough to show that for every $\epsilon > 0$, there is N such that for every $k > m \geq N$ and every z with $|z| \leq 1$ and $|z - 1| \geq \delta$,

$$|f_k(z) - f_m(z)| = |\sum_{n=m+1}^{k} a_n z^n| < \epsilon.$$

Let $B_k = 1 + z + \cdots + z^k$ and $M = \frac{2}{\delta}$. We have $|B_k| \leq \sum_{n=0}^{k} |z^n| = \frac{1-|z|^{k+1}}{1-|z|} \leq M$ for every $k \geq 0$. On the other hand, using summation by part we get

$$\sum_{n=m+1}^{k} a_n z^n = \sum_{n=m+1}^{k-1} B_n(a_n - a_{n+1}) - a_{m+1}B_m + a_k B_k.$$

Since $\lim_{n\to\infty} a_n = 0$, there is N such that for every $k \geq N$, $|a_n| < \frac{\epsilon}{2M}$, so

$$|\sum_{n=m+1}^{k} a_n z^n| \leq M(a_{m+1} - a_k) + a_{m+1}M + a_k M = 2a_{m+1}M < \epsilon.$$

5. The image is the strip $\{z \mid -\frac{\pi}{2} < \text{Im z} < \frac{\pi}{2}\}$.

7. (a) Note that if A and B are such that $a_0 = A + B$ and $a_1 = A\alpha_1 + B\alpha_2$, then by induction, for every $n \geq 2$: $a_n = a_{n-1} + a_{n-2} = A\alpha_1^{n-1} + B\alpha_2^{n-1} + A\alpha_1^{n-2} + B\alpha_2^{n-2} = A\alpha_1^{n-2}(\alpha_1+1)+B\alpha_2^{n-2}(\alpha_2+1)$. But $\alpha_1+1 = \alpha_1^2$ and $\alpha_2+1 = \alpha_2^2$, so $a_n = A\alpha_1^n+B\alpha_2^n$.

So it is enough to find A and B such that

$$A + B = 1$$

and

$$A\alpha_1 + B\alpha_2 = 1$$

which has a unique solution.

(b) Let $\alpha_1 = \frac{1-\sqrt{5}}{2}$ and $\alpha_2 = \frac{1+\sqrt{5}}{2}$. Then

$$|a_n| = |A\alpha_1^n + B\alpha_2^n| = |B|\ |\alpha_2|^n\ |\frac{A}{B}(\frac{\alpha_1}{\alpha_2})^n + 1|.$$

But $|\frac{\alpha_1}{\alpha_2}| < 1$, so $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = |a_2|$, so the radius of convergence is $\frac{1}{\alpha_2}$.

(c) If $|z| < \frac{1}{\alpha_2}$, then

$$\sum_{n=1}^{\infty} a_n z^n = A\sum_{n=1}^{\infty}(\alpha_1 z)^n + B\sum_{n=1}^{\infty}(\alpha_2 z)^n.$$

and both geometric series on the right hand side are convergent (since $|\alpha_1 z| < |\alpha_2 z| < 1$.) The first one converges to $\frac{1}{1-\alpha_1 z}$ and the second one converges to $\frac{1}{1-\alpha_2 z}$. Therefore the $\sum a_n z^n$ converges to

$$\frac{A}{1-\alpha_1 z} + \frac{B}{1-\alpha_2 z} = \frac{(A+B) - z(A\alpha_2 + B\alpha_1)}{1-(\alpha_1+\alpha_2)z + \alpha_1\alpha_2 z^2} = \frac{1}{1-z-z^2}$$

since $\alpha_1 + \alpha_2 = 1, \alpha_1\alpha_2 = -1, A + B = a_0 = 1$, and $A\alpha_2 + B\alpha_1 = (A+B)(\alpha_1+\alpha_2) - A\alpha_1 = B\alpha_2 = 1 - a_1 = 0$.