1. The roots of $z^2 + z + 1$ are the third roots of 1 so each of them is a zero of order 1 and in disk $|z| < 2$. The winding number of $C$ with respect to both roots is 1, so

$$\int_C \frac{(z^2 + z + 1)'}{z^2 + z + 1} \, dz = 4\pi i.$$ 

2. (a) The function $e^z$ maps $\mathbb{C}$ to $\mathbb{C} \setminus \{0\}$ which is not simply connected.
(b) Let $z_0$ be a point in $U$, and for a point $w \in U$, define

$$g(w) = \int_{\gamma} \frac{f'(z)}{f(z)} \, dz$$

where $\gamma$ is any path connecting $z_0$ to $w$ in $U$. This is a well-defined function since the integral does not depend on the choice of the path. By what we have shown before in class, $g$ is holomorphic on $U$ and $g'(z) = \frac{f'(z)}{f(z)}$. So

$$(f(z)e^{-g(z)})' = 0$$

Therefore $f(z)e^{-g(z)}$ is a constant $c$ on $U$. Evaluating at $z_0$, we get $c = f(z_0)$ (since $g(z_0) = 0$). Let $c = e^a$ for some complex number $a_0$, then $f(z)e^{-g(z)} = e^a$, so $f(z) = e^{g(z)+a}$ and $g(z) + a$ is holomorphic on $U$.

3. Near 0, we have $|zf(z)| \leq |z|^{1/2}$. Therefore $\lim_{z \to 0} zf(z) = 0$, so 0 is a removable singularity.

4. To show the statement it is enough to show that if $z_0$ is an isolated singularity of $f(z)$ which is a removable singularity of $e^{f(z)}$, then $z_0$ is a removable singularity of $f(z)$. Assuming this, if $\text{Re } f(z) < M$ for some $M$ and all $z$ near $z_0$, then $|e^{f(z)}| = e^{\text{Re } f(z)} \leq e^M$ and therefore $z_0$ is a removable singularity of $e^f$ and therefore a removable singularity of $f$. Similarly if $\text{Re } f(z) > M$ for some $M$, then
\[ |e^{-f(z)}| = e^{-\text{Re}(f(z))} \leq e^{-M} \] and therefore \( z_0 \) is a removable singularity of \( e^{-f} \) and so a removable singularity of \( -f \) and \( f \).

Assume now that \( z_0 \) is a removable singularity of \( e^f \). Then there is \( M \) such that for all \( z \) near \( z_0 \), \( |e^{f(z)}| < M \), so \( \text{Re}(f(z)) < \log M \), so by Casorati-Weierstrass theorem, \( z_0 \) cannot be an essential singularity of \( f \). If \( z_0 \) is a pole of \( f \), then in a punctured disk near \( z_0 \) we can write \( f(z) = \frac{g(z)}{(z-z_0)^k} \) where \( g \) is holomorphic and \( g(z_0) \neq 0 \). Let \( g(z_0) = re^{i\theta} \), and

\[ z_n = z_0 + \frac{e^{i\theta/n}}{n} \]

Then since \( f(z_n) = g(z_n)e^{-i\theta/n} \) and \( \lim_{n \to \infty} g(z_n) = re^{i\theta} \), \( \lim_{n \to \infty} e^{f(z_n)} = \infty \). So if \( z_0 \) is a pole of \( f \), it cannot be a removable singularity of \( e^f \). This is enough for what we wanted to prove, but you can also show that a pole of \( f \) is never a pole of \( e^f \) by looking at the sequence

\[ w_n = z_0 + \frac{e^{i(\pi+\theta)/n}}{n} \]

and observing that \( \lim_{n \to \infty} e^{f(w_n)} = 0 \).

5. For \( z \) near \( z_0 \) we can write \( f(z) - f(z_0) = a_1(z - z_0) + a_2(z - z_0)^2 + \ldots \) with \( a_1 = f'(z_0) \neq 0 \), so

\[ f(z) - f(z_0) = a_1(z - z_0)(1 + \frac{a_2}{a_1}(z - z_0) + \ldots). \]

So the function \( \frac{z - z_0}{f(z) - f(z_0)} \) has a removable singularity at \( z_0 \) and can be extended to a holomorphic function whose value at \( z_0 \) is the value of the function \( \frac{1}{a_1(1 + \frac{a_2}{a_1}(z - z_0) + \ldots)} \) at \( z_0 \) which is \( \frac{1}{a_1} \). So by the Cauchy’s integral formula, we have

\[ \frac{1}{2\pi i} \int_C \frac{1}{f(z) - f(z_0)} \, dz = \frac{1}{a_1} = \frac{1}{f'(z_0)}. \]