Complex Analysis, Fall 2017

Solutions to Problem Set 6

1. The roots of $z^2 + z + 1$ are the third roots of 1 so each of them is a zero of order 1 and in disk $|z| < 2$. The winding number of C with respect to both roots is 1, so

$$\int_C \frac{(z^2+z+1)'}{z^2+z+1}\,dz = 4\pi i.$$

2.

(a) The function e^z maps $\mathbf{C}$ to $\mathbf{C} \setminus \{0\}$ which is not simply connected.

(b) Let z_0 be a point in U, and for a point $w \in U$, define

$$g(w) = \int_g \frac{f'(z)}{f(z)}\,dz$$

where γ is any path connecting z_0 to w in U. This is a well-defined function since the integral does not depend on the choice of the path. By what we have shown before in class, g is holomorphic on U and $g'(z) = \frac{f'(z)}{f(z)}$. So

$$(f(z)e^{-g(z)})' = 0$$

Therefore $f(z)e^{-g(z)}$ is a constant c on U. Evaluating at z_0, we get $c = f(z_0)$ (since $g(z_0) = 0$). Let $c = e^a$ for some complex number a_0, then $f(z)e^{-g(z)} = e^a$, so $f(z) = e^{g(z)+a}$ and $g(z) + a$ is holomorphic on U.

3. Near 0, we have $|zf(z)| \leq |z|^{1/2}$. Therefore $\lim_{z\to 0} zf(z) = 0$, so 0 is a removable singularity.

4. To show the statement it is enough to show that if z_0 is an isolated singularity of $f(z)$ which is a removable singularity of $e^{f(z)}$, then z_0 is a removable singularity of $f(z)$. Assuming this, if $\mathrm{Re\,f(z)} < M$ for some M and all z near z_0, then $|e^{f(z)}| = e^{\mathrm{Re\,f(z)}} \leq e^M$ and therefore z_0 is a removable singularity of e^f and therefore a removable singularity of f. Similarly if $\mathrm{Re\,f(z)} > M$ for some M, then

$|e^{-f(z)}| = e^{-\text{Re f(z)}} \leq e^{-M}$ and therefore z_0 is a removable singularity of e^{-f} and so a removable singularity of $-f$ and f.

Assume now that z_0 is a removable singularity of e^f. Then there is M such that for all z near z_0, $|e^{f(z)}| < M$, so Re f(z) $< \log M$, so by Casorati-Weierstrass theorem, z_0 cannot be an essential singularity of f. If z_0 is a pole of f, then in a punctured disk near z_0 we can write $f(z) = \frac{g(z)}{(z-z_0)^k}$ where g is holomorphic and $g(z_0) \neq 0$. Let $g(z_0) = re^{i\theta}$, and

$$z_n = z_0 + \frac{e^{i\theta/n}}{n}$$

Then since $f(z_n) = g(z_n)e^{-i\theta}n^n$ and $\lim_{n\to\infty} g(z_n) = re^{i\theta}$, $\lim_{n\to\infty} e^{f(z_n)} = \infty$. So if z_0 is a pole of f, it cannot be a removable singularity of e^f. This is enough for what we wanted to prove, but you can also show that a pole of f is never a pole of e^f by looking at the sequence

$$w_n = z_0 + \frac{e^{i(\pi+\theta)}/n}{n}$$

and observing that $\lim_{n\to\infty} e^{f(w_n)} = 0$.

5. For z near z_0 we can write $f(z) - f(z_0) = a_1(z - z_0) + a_2(z - z_0)^2 + ...$ with $a_1 = f'(z_0) \neq 0$, so

$$f(z) - f(z_0) = a_1(z - z_0)(1 + \frac{a_2}{a_1}(z - z_0) + ...).$$

So the function $\frac{z-z_0}{f(z)-f(z_0)}$ has a removable singularity at z_0 and can be extended to a holomorphic function whose value at z_0 is the value of the function $\frac{1}{a_1(1+\frac{a_2}{a_1}(z-z_0)+...)}$ at z_0 which is $\frac{1}{a_1}$. So by the Cauchy's integral formula, we have

$$\frac{1}{2\pi i}\int_C \frac{1}{f(z) - f(z_0)}\, dz = \frac{1}{a_1} = \frac{1}{f'(z_0)}.$$