1. Let $K$ be a compact subset of $\mathbb{C}$ and $S$ the set of continuous functions on $U$ which can be approximated uniformly on $K$ by polynomials.

(a) Show that if $f, g \in S$, then $f + g \in S$ and $gf \in S$.

(b) Show that if $f_n \in S$ and $f$ is a function on $U$ such that $f_n$ converges to $f$ uniformly on $K$, then $f \in S$.

2. Let $U \subset \mathbb{C}$ be a non-empty open set. Show that there is a sequence of compact subsets $K_n$ of $U$ such that $U = \bigcup_n K_n$ and $K_n$ is contained in the interior of $K_{n+1}$ for every $n \geq 1$. (Let $K_n$ be the set of point $z$ in $U$ such that $|z| \leq n$ and the distance between $z$ and the boundary of $U$ is $\geq \frac{1}{n}$.)

3. Use Problem 2 to show that if $U \subset \mathbb{C}$ is an open subset, $\{a_n\}_{n=1}^\infty$ is a sequence of distinct points of $U$ without a limit point in $U$, and $p_n$ polynomials whose constant coefficients are zero, then there is a meromorphic function $f$ on $U$ such that the poles of $f$ are exactly the $a_n$ and the principal part of $f$ at $a_n$ is $p_n(\frac{1}{z-a_n})$.

4. Show that if $f$ and $g$ are entire functions with no common zeros, then there are entire functions $r$ and $s$ such that $fr + gs = 1$. (Such $f$ and $g$ are called relatively prime.) Hint: Write $\frac{1}{fg} = F + G$ where $F$ only has poles at zeros of $f$ and $G$ has poles only at zeros of $g$.

5. Let $U \subset \mathbb{C}$ be an open subset, and let $K_n, n \geq 1$, be as in Problem 2. For continuous functions $f, g$ on $U$, we define $d(f, g)$ as follows: For $n \geq 1$, let

$$
\delta_n(f, g) = \max_{z \in K_n} \frac{|f(z) - g(z)|}{1 + |f(z) - g(z)|},
$$
and

\[ d(f, g) = \sum_{n=1}^{\infty} 2^{-n} \delta_n(f, g). \]

(a) Show that \( d \) is a metric on \( C(U) \), the space of continuous functions on \( U \).

(b) Show that \( f_n \to f \) in \( (C(U), d) \) if and only if \( f_n \) converges uniformly to \( f \) on all compact subsets of \( U \).