Algebra I

Problem Set 1

Due: September 13 in class

1. Let N be a normal subgroup of a group G. For a subgroup H in G, let

$$\phi(H) = \{hN | h \in H\} \subset G/N.$$

   (a) Show ϕ gives a 1-1 correspondence between subgroups of G which contain N and subgroups of G/N.

   (b) Show that if $N \leq H_1 \leq H_2$, then $H_1 \trianglelefteq H_2$ if and only if $H_1/N \trianglelefteq H_2/N$, and in this case

$$H_2/H_1 \simeq (H_2/N)/(H_1/N).$$

2. (a) Show that A_5 is simple. (b) Find all the normal subgroups of S_n when $n \geq 5$.

3. If $H \leq K \leq G$, then show that $[G : H] = [G : K][K : H]$. (Do not assume G is finite.)

4. Show that if $n \geq 2$, S_n is generated by the two cycles $\tau = (1\ 2)$ and $\sigma = (1\ 2\ \ldots n)$. (Hint: $(i\ i+1) = \sigma(i-1\ i)\sigma^{-1}$)

5. Let G be the subgroup of $GL(2, \mathbf{C})$ generated by

$$A = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

   (a) Show G is of order 8 and is not abelian.

   (b) Show that every subgroup of G is normal.

6. Soppose that G is a finite group, $H \leq G$ and $N \trianglelefteq G$ such that $|H|$ and $[G : N]$ are relatively prime. Show that $H \leq N$.

7. Let G be a group. For x, y in G the *commutator* of x and y is

$$[x, y] = xyx^{-1}y^{-1}.$$

Let G' be the subgroup of G generated by all the commutators.

- Show $[x, y]^{-1} = [y, x]$ and $G' = \{[x_1, y_1] \dots [x_m, y_m] | \ x_i, y_i \in G\}$.
- Show $G' \trianglelefteq G$ and G/G' is abelian.
- Set $G^{(0)} = G$, and $G^{(i)} = (G^{(i-1)})'$ for $i \geq 1$. Show G is solvable if and only if $G^{(m)} = \{e\}$ for some $m \geq 0$.