Algebra I

Problem Set 2

Due: September 20 in class

1. If N, H_1, H_2 are subgroups of a group G such that $N \trianglelefteq G$ and $H_1 \trianglelefteq H_2$, then show

$$NH_1 \trianglelefteq NH_2.$$

2. If H_1 and H_2 are subgroups of a group G, then show

$$|H_1 H_2| = \frac{|H_1|\,|H_2|}{|H_1 \cap H_2|}.$$

3. (Lang, Ex. 1.16) Let H be a proper subgroup of a finite group G. Prove that G is not the union of all conjugates of H.

4. (Lang, Ex. 1.20) If $|G| = p^n$ where p is a prime number, then show that every normal subgroup of G which has order p is contained in the center of G.

5. Let X be a finite set with $|X| \geq 2$ and G a finite group acting on X. Show that if the action is transitive, then there is an element $g \in G$ that does not fix any element of X: $gx \neq x$ for all $x \in X$.

6. Assume G is a finite group. Show that there is an injective group homomorphism $G \to S_G$, where S_G is the group of permutations of G.

7. Show if G is a finite group and Aut(G) is cyclic, then G is abelian.