Algebra I, Fall 2016

Problem Set 10

Due: December 8 in class

1. Let K be a field, and let $K(t)$ be the field of rational functions in one variable t over K. Show that if an element $\alpha \in K(t)$ is algebraic over K, then $\alpha \in K$.

2. Find the splitting fields of the following polynomials over $\mathbf{Q}$, and find the degree of such splitting fields over $\mathbf{Q}$.

(i) $x^2 - 2$

(ii) $x^3 - 2$

(iii) $x^2 + x + 1$

3. Let K be a finite or countable field. Show that the algebraic closure of K has countably many elements.

4. Let F be a field of characteristic p (p a prime number) such that every polynomial in $F[x]$ is separable. Show that for every $a \in F$, there is $b \in F$ such that $a = b^p$. (Hint: look at the polynomial $f(x) = x^p - a$. Show that if there is no such b, then $f(x)$ is irreducible and nonseparable.)