Algebra I

Problem Set 3

Due: September 27 in class

1. Show that every group G with $|G| < 60$ is solvable (we proved the statement for $|G| \leq 30$ in class.)

2. Show that if $|G| = pqr$ where p, q, r are distinct prime numbers, then G has a normal Sylow subgroup.

3. Let G_1 and G_2 be finite groups, and let p be a prime number. Show that every p-Sylow subgroup of $G_1 \times G_2$ is of the form $P_1 \times P_2$ where P_1 is a p-Sylow subgroup of G_1 and P_2 is a p-Sylow subgroup of G_2.

4. Let G and H be two groups, and let $\phi : H \to \mathrm{Aut(G)}$ be a homomorphisms. For $h \in H$, we denote the image of h under ϕ by ϕ_h. We define an operation on the cartesian product of G and H as follows:

$$(g_1, h_1)(g_2, h_2) = (g_1\phi_{h_1}(g_2), h_1h_2).$$

(a) Show that the cartesian product is a group under this operation. This group is called the *semidirect product* of G and H and is denote by $G \rtimes H$.

(b) Show that the natural map from G to $G \rtimes H$ sending g to (g, e_H) is a group homomorphism and its image is a normal subgroup of $G \rtimes H$.

(c) If ϕ is trivial (so $\phi_h(g) = g$ for all h, g), then the semidirect product is the direct product $G \times H$.

5. The group $\mathbf{Z}/n$ admits an automorphism of order 2 sending $[k]$ to $[-k]$. So there is a group homomorphisms $\phi : \mathbf{Z}/2 \to \mathrm{Aut}(\mathbf{Z}/\mathrm{n})$. Show that the resulting semidirect product is isomorphic to the Dihedral group of order $2n$.

6. Let G be a group with subgroups H and K such that $K \cap H = \{e\}$, and $KH = G$.

   (a) Show that if K is normal, then G is isomorphic to $K \rtimes H$, where $\phi_h(k) = hkh^{-1}$.

   (b) Show that if K and H are both normal, then G is isomorphic to $K \times H$.