Algebra I, Fall 2016

Problem Set 4

Due: October 6 in class

1. The group $\mathbf{Z}$ acts on $\bigoplus_{n\in\mathbf{Z}}\mathbf{Z}_2$ (a direct sum of copies of $\mathbf{Z}_2$ indexed by $\mathbf{Z}$) by shifting (so there is a group homomorphism $\phi : \mathbf{Z} \to \mathrm{Aut}(\bigoplus_{\mathrm{n}\in\mathbf{Z}}\mathbf{Z}_2)$ such that for $S \in \bigoplus_{n\in\mathbf{Z}}\mathbf{Z}_2$, the i-th component of $\phi(m)(S)$ is the $(i-m)$-th component of S.) Show the resulting semidirect product

$$(\bigoplus_{n\in\mathbf{Z}}\mathbf{Z}_2) \rtimes \mathbf{Z}$$

is finitely generated. Note that we have seen in Homework 3, Question 4(b), that this group has a subgroup isomorphic to $\bigoplus_{n\in\mathbf{Z}}\mathbf{Z}_2$, so there is a subgroup which is not finitely generated.

2. (Lang I.42) Show that $\mathbf{Q}/\mathbf{Z}$ (as an additive group) has one and only one subgroup of order n for each integer $n \geq 1$, and that this subgroup is cyclic.

3. (Lang I.43) Let H be a subgroup of a finite abelian group G. Show that G has a subgroup that is isomorphic to G/H.

4. Let A be a finite abelian group, and let p be a prime number. Show that the number of subgroups of order p in A equals the number of subgroups of index p in A.

5. Let p be a prime number and let F be a field with p-elements. Find a p-Sylow subgroup of $GL(2, F)$.

6. If p is a prime number, then show every group of order $2p$ is either cyclic or isomorphic to the Dihedral group of order $2p$.