Algebra I, Fall 2016

Problem Set 5

Due: October 13 in class

In all the following questions, R and S are assumed to be a commutative ring.

1. If $f : R \to S$ is a ring homomorphism. Prove the following.

   (a) If P is a prime ideal of S, then $f^{-1}(P)$ is either equal to R a prime ideal of R.

   (b) If P is a maximal ideal of S, then $f^{-1}(P)$ is not necessarily a maximal ideal of R.

2. For an ideal I of R, let

$$\sqrt{I} = \{x \in R \mid x^n \in I \text{ for some } n \geq 1\}.$$

$\sqrt{I}$ is called the *radical* of I.

   (a) Show that $\sqrt{I}$ is an ideal of R which contains I.

   (b) Show that $\sqrt{IJ} = \sqrt{I \cap J}$ for any two ideals I and J.

   (c) Give an example such that $IJ \neq I \cap J$.

3. Let S a multiplicative subset of R not containing 0, and let $\phi : R \to S^{-1}R$ be the map $\phi(r) = \frac{r}{1}$. For an ideal I in R, let

$$S^{-1}I = \{\frac{i}{s} \mid i \in I, s \in S\} \subset S^{-1}R.$$

   (a) Show that $S^{-1}I$ is an ideal of $S^{-1}R$, and $S^{-1}\ \phi^{-1}(J) = J$ for any ideal J of $S^{-1}R$.

(b) Show the map $P \mapsto S^{-1}P$ gives a one-to-one correspondence between prime ideals of R whose intersection with S is empty and prime ideals of $S^{-1}R$.

4. Show that in every finite commutative ring, every prime ideal is maximal.

5. (Lang II.4, Question 1) Let S be a multiplicative subset of R not containing 0. Let P be a maximal element in the set of ideals of R whose intersection with S is empty. Show that P is a prime idea.

6. A proper ideal I of R is said the be a *primary ideal* if $ab \in I$ implies $a \in I$ or $b^n \in I$ for some positive integer n.

(a) Find all the primary ideals of $\mathbf{Z}$.

(b) Show that if I is a primary ideal, then $\sqrt{I}$ is a prime ideal.