Algebra I, Fall 2016

Problem Set 7

Due: November 10 in class

1. Using Eisenstein critetrion show that the polynomials x^4+1 and x^6+x^3+1 are irreducible over the field of rational numbers (Hint: change x to $x+1$).

2. Dual Modules: Let R be a commutative ring, and let M be a module over R. Define the *dual* of M denoted by $M^\vee = \mathrm{Hom_R(M,R)}$ to be the set of all R-module homormorphisms from M to R.

(i) Show that $M^\vee$ is an R-module.

(ii) Show that $(M \oplus N)^\vee$ is isomorphic to $M^\vee \oplus N^\vee$.

3. (Lang III, 9) Let R be a commutative ring, and let M be a R-module. Let S be a multiplicative subset of R such that $1 \in S$ and $0 \notin S$. Consider the set of all $\{(m,s), m \in M, s \in S\}$, and show that the relation

$$(m_1, s_1) \sim (m_2, s_2) \text{ if there is } s \in S \text{ such that } s(s_2 m_1 - s_1 m_2) = 0$$

is an equivalence relation. Denote the class of (m,s) by $\frac{m}{s}$, and set

$$S^{-1}M = \{(m,s), m \in M, s \in S\}/\sim$$

(i) Show that $S^{-1}M$ is a module over $S^{-1}R$.

(ii) If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of R-modules, show that $0 \to S^{-1}M' \to S^{-1}M \to S^{-1}M'' \to 0$ is an exact sequence of $S^{-1}M$-modules.

4. (Lang III, 15) **The five lemma.** Let R be a commutative ring, and consider a

commutative diagram of R-modules such that every row is exact

$$\begin{array}{ccccccccc}
M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & M_4 & \longrightarrow & M_5 \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\
N_1 & \longrightarrow & N_2 & \longrightarrow & N_3 & \longrightarrow & N_4 & \longrightarrow & N_5.
\end{array}$$

(i) If f_1 is surjective and f_2 and f_4 are injective, then f_3 is injective.

(ii) If f_5 is injective andf_2 and f_4 are surjective, then f_3 is surjective.