Algebra I, Fall 2016

Problem Set 8

Due: November 17 in class

In all questions R is a commutative ring.

1. Let

$$M' \to M \to M'' \to 0$$

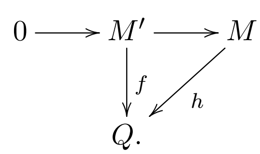
be a sequence of R-modules. Show that if

$$0 \to \mathrm{Hom(M'', P)} \to \mathrm{Hom(M, P)} \to \mathrm{Hom(M', P)}$$

is exact for every R-module P, then the sequence itself is exact.

2. Let R be a commutative ring and Q a module over R. Show that the following are equivalent:

(a) If M is an R-module, if M' is a submodule of M, and if $f : M' \to Q$ is a R-homomorphism, then there is an extension of f to a R-homomorphism $M \to Q$, i.e., there is a R-homomorphism $h : M \to Q$ such that the following diagram is commutative



(b) For any short exact sequence $0 \to M' \to M \to M'' \to 0$, the sequence

$$0 \to \mathrm{Hom_R(M'', Q)} \to \mathrm{Hom_R(M, Q)} \to \mathrm{Hom_R(M', Q)} \to 0$$

is exact.

(c) Every short exact sequence $0 \to Q \to M \to M'' \to 0$ splits.

If the above equivalent conditions are satisfied, the module Q is called an *injective* module.

3. Use Zorn's lemma to show that $\mathbf{Q}$ is an injective $\mathbf{Z}$-module. (Hint: Let $N \to M$ be an injective homomorphism of abelian groups, and let $g : \mathbf{Q} \to N$ be a group homomorphism. Identifying N with its image, we can consider N as subgroup of M. Consider now the set of all pairs (H, ϕ) where H is a subgroup of M containing N and $\phi : \mathbf{Q} \to H$ is a group homomorphism extending g. Show that this set has a maximal element, and that the maximal element has to be of the form (M, ϕ) for some ϕ.)

4. (a) Let I, J be ideals of R. Show

$$R/I \otimes_R R/J \cong R/(I + J).$$

(b) Show $\mathbf{Z}_n \otimes_{\mathbf{Z}} \mathbf{Z}_m \cong \mathbf{Z}_d$ where $d = gcd(m, n)$.