Algebra I, Fall 2016

Problem Set 9

Due: December 1 in class

In all questions R is a commutative ring.

1. Let M be a R-module and $I \subset R$ an ideal. Show that

$$\frac{M}{IM} \simeq M \otimes_R \frac{R}{I}$$

as $\frac{R}{I}$-modules.

2. Let R be a PID, and p a prime element in R. Assume that M is a finitely generated R-module such that for every $m \in M$, $p^s m = 0$ for some $s \geq 0$.

(a) Let $M =< m_1, \ldots, m_k >$ and let s_i be the smallest integer such that $p^{s_i} m_i = 0$. Assume $s_1 \geq \cdots \geq s_k$. Let $N =< m_1 >$. Show that every element $x + N$ of M/N can be lifted to an element $y \in M$ (so $y + N = x + N$) with the property that $p^r y = 0$ if and only if $p^r(y + N) = 0$.

(b) Show that M can be written as a direct sum

$$M \simeq M_1 \oplus \cdots \oplus M_n$$

of $R-$modules such that each M_i is generated by one element. (Hint: Use induction on the number of a set of generators and follow the argument in class).

3. Let R be a PID, and let M be a finitely generated torsion R-module. For $a \in R$, set

$$M_a = \{m \in M | am = 0\}.$$

(i) Show that if $m \in M$, and if $am = 0$ and $bm = 0$, then $\gcd(a, b)m = 0$. (use the fact that $\gcd(a, b)$ can be written as a linear combination of a and b.)

(ii) Show that If $a = bc$ and $\gcd(b, c) = 1$, then

$$M_a = M_b \oplus M_c.$$

(iii) Let $p \in R$ be a prime element. If $m \in M_{p^n}$, $n \geq 1$, then show that $< m >$, the sudmodule generated by m, is isomorphic to $R/(p^s)$ for some $s \leq n$, where (p^s) is the ideal generated by p^s.

(iv) Use parts (i)-(iii) and Question 2 to show that

$$M \simeq R/(p_1^{n_1}) \oplus \cdots \oplus R/(p_s^{n_s})$$

for some prime elements $p_i \in R$ and positive integers n_i.

4. Let F be a field and $E = F(\alpha)$ where α is algebraic of odd degree over F. (So $[F(\alpha) : F]$ is odd.) Show that $E = F(\alpha^2)$.