Algebra I, Fall 2016

Solutions to Problem Set 7

1. It is enough to show $f(x+1) = (x+1)^4 + 1$ and $g(x+1) = (x+1)^6 + (x+1)^3 + 1$ are irreducible. But $f(x+1) = x^4 + 4x^3 + 6x^2 + 4x + 2$, and $g(x+1) = x^6 + 6x^5 + 15x^4 + 21x^3 + 18x^2 + 9x + 3$, so $p = 2$ works for f and $p = 3$ works for g when we apply the Eisenstein irreducibility criterion.

2. (i) The module structure is given by $(f+g)(m) = f(m) + g(m)$, and $(rf)(m) = rf(m)$ for every $r \in R$ and $f, g \in \mathrm{Hom(M, R)}$.

(ii) We define a R-homomorphisms

$$\phi : M^\vee \oplus N^\vee \to (M \oplus N)^\vee$$

as follows. If $(f, g) \in M^\vee \oplus N^\vee$, then $\phi(f, g)(m, n) = f(m) + g(n)$. Clearly ϕ is a R-homomorphism.

- ϕ is injective: if $\phi(f, g) = 0$, then $\phi(f, g)(m, 0) = 0$ for all $m \in M$, so $f(m) = 0$ for all $m \in M$, so $f = 0$, and similarly $g = 0$, so $(f, g) = 0$.
- ϕ is surjective: if $h \in (M \oplus N)^\vee$ is given, then let $g \in M^\vee$ be defined by $f(m) = h(m, 0)$, and let $g \in N^\vee$ be defined by $g(n) = h(0, n)$. Then $\phi(f, g)(m, n) = f(m) + g(n) = h(m, 0) + h(0, n) = h(m, n)$.

3. (i) The operations are given by $\frac{m_1}{s_1} + \frac{m_2}{s_2} := \frac{s_2 m_1 + s_1 m_2}{s_1 s_2}$, and $\frac{r}{s} \cdot \frac{m}{s'} := \frac{rm}{ss'}$. (It is easy to see that these are well-defined, i.e. they don't depend on the element representing a class.)

(ii) Consider the sequences

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

and

$$0 \longrightarrow S^{-1}M' \xrightarrow{\tilde{f}} S^{-1}M \xrightarrow{\tilde{g}} S^{-1}M'' \longrightarrow 0$$

Note that $\tilde{f}$ is given by $\tilde{f}(\frac{m}{s}) = \frac{f(m)}{s}$, and $\tilde{g}$ is similarly defined.

- $\tilde{g}$ is surjective: if $\frac{x}{s} \in S^{-1}M''$ is given, then there is $y \in M$ such that $g(y) = x$, so $\tilde{g}(\frac{y}{s}) = \frac{x}{s}$.

- $\tilde{f}$ is injective: if $\tilde{f}(\frac{x}{s}) = 0$, then $\frac{f(x)}{s} = 0$, so there is $s' \in S$ such that $s'f(x) = 0$. So $f(s'x) = 0$. Since f is injective, this implies $s'x = 0$, so $\frac{x}{s} = 0$.

- $\tilde{g} \circ \tilde{f} = 0$: we have $\tilde{g} \circ \tilde{f}(\frac{x}{s}) = \tilde{g}(\frac{f(x)}{s}) = \frac{g(f(x))}{s} = 0$.

- kernel $(\tilde{g}) \subset$ Image $(\tilde{f})$: if $\tilde{g}(\frac{x}{s}) = 0$, then $\frac{g(x)}{s} = 0$, so there is $s' \in S$ such that $s'g(x) = 0$, so $g(s'x) = 0$, so there is $y \in M'$ such that $f(y) = s'x$, so $\tilde{f}(\frac{y}{s's}) = \frac{s'x}{s's} = \frac{x}{s}$.

4.

$$\begin{array}{ccccccccc} M_1 & \xrightarrow{g_1} & M_2 & \xrightarrow{g_2} & M_3 & \xrightarrow{g_3} & M_4 & \xrightarrow{g_4} & M_5 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\ N_1 & \xrightarrow{h_1} & N_2 & \xrightarrow{h_2} & N_3 & \xrightarrow{h_3} & N_4 & \xrightarrow{h_4} & N_5. \end{array}$$

(i). Let $x \in M_3$ be such that $f_3(x) = 0$. Then $f_4 \circ g_3(x) = h_3 \circ f_3(x) = 0$. Since f_4 is injective, this implies that $g_3(x) = 0$. Since the top sequence is exact, there should be $y \in M_2$ such that $g_2(y) = x$. Therefore, $h_2 \circ f_2(y) = f_3 \circ g_2(y) = f_3(x) = 0$. Let $t = f_2(y)$. Then $h_2(t) = 0$, and since the lower sequence is exact, this implies that there is $s \in N_1$ such that $h_1(s) = t$. But f_1 is surjective, so there is $m \in M_1$ such that $f_1(m) = s$. Now $f_2 \circ g_1(m) = h_1 \circ f_1(m) = h_1(s) = t$, Since f_2 is injective and $f_2(g_1(m)) = f_2(y) = t$, we conclude that $g_1(m) = y$ and therefore $x = g_2 \circ g_1(m) = 0$ since the top sequence is exact at M_2.

(ii) Assume $x \in N_3$ if given. Let $t = h_3(x)$. Since the bottom sequence is exact at N_4, we have $h_4(t) = 0$. Since f_4 is surjective, there is $s \in M_4$ such that $f_4(s) = t$. We have $f_5 \circ g_4(s) = h_4 \circ f_4(s) = h_4(t) = 0$. Since f_5 is injective, this implies $g_4(s) = 0$. Since the top sequence is exact at M_4 this implies there is $y \in M_3$ such that $g_3(y) = s$. Let $x' = f_3(y)$. Then $h_3(x') = h_3 \circ f_3(y) = f_4 \circ g_3(y) = f_4(s) = t$. So $h_3(x - x') = t - t = 0$. Therefore there is $a \in N_2$ such that $h_2(a) = x - x'$. Since f_2 is surjective, there is $b \in M_2$ such that $f_2(b) = a$. So $f_3 \circ g_2(b) = h_2 \circ f_2(b) = h_2(a) = x - x'$. So $f_3(y + g_2(b)) = f_3(y) + f_3(g_2(b)) = x' + (x - x') = x$. So x is in the image of f_3.