Algebra I, Fall 2016

Solutions to Problem Set 8

1. We have the sequences

$$M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0.$$

$$0 \longrightarrow \mathrm{Hom(M'', P)} \xrightarrow{\phi} \mathrm{Hom(M, P)} \xrightarrow{\psi} \mathrm{Hom(M', P)}$$

where ϕ is composition with g and ψ is composition with f.

- g is surjective: Let $P = M''/Im(g)$ and $h : M'' \to P$ the canonical map. Then $h \circ g = 0$. Since ϕ is injective, and since $\phi(h) = h \circ g = 0$, h should be the zero map. So $M'' = Im(g)$.

- $Im(f) \subset Ker(g)$: We know $\psi \circ \phi = 0$ and want to show $g \circ f = 0$. Let $P = M''$ and $i \in Hom(M'', P)$ the identity map. Then $0 = \psi \circ \phi(i) = i \circ g \circ f = g \circ f$.

- $Ker(g) \subset Im(f)$: Let $P = M/Im(f)$, and $h : M \to P$ the canonical homomorphism. Then $h \circ f = 0$, so $\psi(h) = 0$, so by the exactness of the hom sequence, there is $q \in Hom(M'', P)$ such that $\phi(q) = h$. That means $q \circ g = h$. So if $g(x) = 0$, then $h(x) = q(g(x)) = 0$, so by definition of h, x should be in the image of f.

2. Clearly (a) implies (b) since the sequence

$$0 \to \mathrm{Hom_R(M'', Q)} \to \mathrm{Hom_R(M, Q)} \to \mathrm{Hom_R(M', Q)}$$

is always exact, and (a) says the map $\mathrm{Hom_R(M, Q)} \to \mathrm{Hom_R(M', Q)}$ is surjective.

To show (b) implies (c), set $M' = Q$. Then by (b), the map $\mathrm{Hom_R(M, Q)} \to \mathrm{Hom_R(Q, Q)}$ is surjective. So the identity map $i : Q \to Q$ should be the image of a map $h : M \to Q$. (so the composition of h with the injective map $Q \to M$ is the identity map). So by definition the sequence splits.

To show (c) implies (a) let $\alpha : M' \to M$ be an injective R-homomorphism and $f : M' \to Q$ a R-homomorphism. Define a R-homomorphism $\beta : M' \to Q \oplus M$ by $\beta(x) = (-f(x), \alpha(x))$. The image of β is a submodule of $Q \oplus M$. Let $N = (Q \oplus M)/Im(\beta)$. There are two R-homomorphisms $\tilde{\alpha} : Q \to N$, sending q to $(q, 0) + Im(\beta)$ and $\tilde{f} : M \to N$ sending m to $(0, m) + Im(\beta)$.

$$\begin{array}{ccc} M' & \xrightarrow{\alpha} & M \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \tilde{f}} \\ Q & \xrightarrow{\tilde{\alpha}} & N \end{array}$$

We have:

- $\tilde{f} \circ \alpha = \tilde{\alpha} \circ f$: $\tilde{f} \circ \alpha(x) = (0, \alpha(x)) + Im(\beta)$ and $\tilde{\alpha} \circ f(x) = (f(x), 0) + Im(\beta)$. But $(f(x), 0) - (0, \alpha(x)) = \beta(x) \in Im(\beta)$

- $\tilde{\alpha}$ is injective: if $\tilde{\alpha}(q) = 0$, then $(q, 0) \in Im(\beta)$, so $(q, 0) = (-f(x), \alpha(x))$ for some $x \in M'$. So $\alpha(x) = 0$, but α is injective, so $x = 0$ and $q = -f(0) = 0$.

- by part (c) there is a r-homomorphism $\gamma : N \to Q$ such that $q \circ \tilde{\alpha} = id_Q$

- $\gamma \circ \tilde{f} : M \to Q$ is the extension of f to M. (i.e. $f = (\gamma \circ \tilde{f}) \circ \alpha$)

3. Consider the set of pairs (H, ϕ) such that H is a subgroup of M and $\phi : H \to \mathbf{Q}$ extends g. By Zorn's lemma, there is a maximal element (H_0, ϕ_0) in this set. If $H_0 = M$, we are done. Otherwise pick $m \in M \setminus H$. We get a contradiction by showing it is possible to extend g to $< m, H >$. Let $G = \{a \in \mathbf{Z} | am \in H\}$. Clearly G is a subgroup of $\mathbf{Z}$ and therefore generated by an integer a_0. If $a_0 = 0$, then $< m > \cap H = \emptyset$, and we define $\phi :< m, H >\to \mathbf{Q}$ by $\phi(bm + h) = \phi_0(h)$. This is clearly well-defined.

If $a_0 \neq 0$, then we define $\phi : H \to \mathbf{Q}$ by

$$\phi(bm + h) = \frac{b}{a_0} q + \phi_0(h) \quad b \in \mathbf{Z}, h \in H$$

This is well defined: if $bm + h = b'm + h'$, then $(b - b')m = h' - h \in H$, so $a_0 | b - b'$, so $b - b' = ca_0$, and so $\phi_0(h' - h) = \phi_0((b - b')m) = \phi_0(c(a_0 m)) = cq = \frac{b - b'}{a_0} q$. Therefore, $\frac{b}{a_0} q + \phi_0(h) = \frac{b'}{a_0} q + \phi_0(h')$.

4. We defined a map $R/I \times R/J$ by sending $(r + I, s + J)$ to $rs + (I + J)$. This is well defined (if $r - r' \in I$, then $(r - r')s \in I$, so $(r - r')s \in I + J$, and similarly if

$s - s' \in J$, then $r(s-s') \in I+J$) and clearly bilinear, so it induces a R-homomorphism $\phi : R/I \otimes_R R/J \to R/(I+J)$.

We can also defined a R-homomorphism $\psi : R/(I+J) \to R/I \otimes_R R/J$ by

$$\psi(r + (I+J)) = (r+I) \otimes (1+J).$$

Note that this is well defined: if $r + (I+J) = 0$, then $r \in (I+J)$, so $r = i + j$, $i \in I, j \in J$, so $(r+I) \otimes (1+J) = (j+I) \otimes (1+J) = j((1+I) \otimes (1+J)) = (1+I) \otimes (j+J) = (i+I) \otimes 0 = 0$. It is easy to see ϕ and ψ are inverse to each other.