1. What is the angle between the two planes $x + y = 1$ and $2x + y - 2z = 2$?

(a) $\frac{\pi}{2}$

(b) $\frac{\pi}{3}$

(c) $\frac{\pi}{6}$

(d) $\frac{\pi}{4}$

(e) The two planes are parallel

(f) $\frac{2\pi}{3}$

$\vec{n}_1 = \langle 1, 1, 0 \rangle \qquad \vec{n}_2 = \langle 2, 1, -2 \rangle$

$|\vec{n}_1| = \sqrt{2} \qquad |\vec{n}_2| = 3 \qquad \vec{n}_1 \circ \vec{n}_2 = 3$

$\Rightarrow \cos\theta = \dfrac{\vec{n}_1 \circ \vec{n}_2}{|\vec{n}_1|\,|\vec{n}_2|} = \dfrac{\sqrt{2}}{2}$

2. If $\mathbf{u} = \langle -4, -1, -2 \rangle$ and $\mathbf{v} = \langle 2, 2, 1 \rangle$, then what is $|\mathbf{u} \times \mathbf{v}|$?

(a) 6

(b) 9

(c) 15

(d) $3\sqrt{2}$

(e) $3\sqrt{3}$

(f) $3\sqrt{5}$

$\vec{u} \times \vec{v} = \langle 3, 0, -6 \rangle$

So $|\vec{u} \times \vec{v}| = \sqrt{9 + 36} = \sqrt{45} = \cancel{3\sqrt{6}}\ 3\sqrt{5}$

3. The rate of change of f at the point (a, b) in the direction of $\mathbf{i}$ is 2 and in the direction of $\dfrac{\mathbf{i}}{\sqrt{2}} + \dfrac{\mathbf{j}}{\sqrt{2}}$ is $\sqrt{2}$. In which direction is the rate of change of f at (a, b) minimum?

(a) $\langle 1,\ 0\rangle$

(b) $\langle -1,\ 0\rangle$

(c) $\langle \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\rangle$

(d) $\langle \frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\rangle$

(e) $\langle 0,\ 1\rangle$

(f) $\langle 0,\ -1\rangle$

$f_x = 2$ and $\langle f_x, f_y\rangle \cdot \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle = \sqrt{2} \Rightarrow f_x + f_y = 2 \Rightarrow f_y = 0$

So $\nabla f\,(a,b) = \langle 2, 0\rangle$

4. Which of the following is an equation of the tangent line to the curve of intersection of surfaces $xyz = 2$ and $3x^2 + y^2 - z^2 = 0$ at the point $(1, 1, 2)$?

(a) $x = 1 - 5t, \quad y = 1 + 7t, \quad z = 2 - 4t$

(b) $x = 1 + 5t, \quad y = 1 + 7t, \quad z = 2 + t$

(c) $x = 1 - t, \quad y = 1 - 3t, \quad z = 2 + 4t$

(d) $x = 1 - t, \quad y = 1 + 3t, \quad z = 2 + t$

(e) $x = 1 - 5t, \quad y = 1 - 3t, \quad z = 2 + t$

(f) $x = 1 - 5t, \quad y = 1 + 3t, \quad z = 2 - 2t$

$f(x,y,z) = xyz \qquad \nabla f = \langle yz, xz, xy \rangle \Rightarrow \nabla f(1,1,2) = \langle 2,2,1 \rangle$

$g(x,y,z) = 3x^2 + y^2 - z^2 \qquad \nabla g = \langle 6x, 2y, -2z \rangle \Rightarrow \nabla g(1,1,2) = \langle 6,2,-4 \rangle$

$\langle 2,2,1 \rangle \times \langle 6,2,-4 \rangle = \langle -10, 14, -8 \rangle$

5. If $w = z - \sin(xy)$, $x = t$, $y = \ln t$, and z is a function of t such that $z'(1) = 2$, then what is $\frac{dw}{dt}$ at $t = 1$?

(a) -2

(b) -1

(c) 0

(d) 1

(e) 2

(f) $\frac{1}{2}$

$$\frac{dw}{dt} = \frac{\partial w}{\partial z}\frac{dz}{dt} + \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

$$= z'(t) - y\cos(xy)\,x'(t) - x\cos(xy)\,y'(t)$$

$t=1 \Rightarrow y = \ln 1 = 0 \quad x = 1 \quad x'(1) = 1 \quad y'(1) = 1$

$$\Rightarrow \frac{dw}{dt}(1) = z'(1) - 0\,\cancel{\cos 0} - \cos 0 = 2 - 1 = 1$$

6. If $f_x = 2x - 4y$ and $f_y = 2y - 4x$, then which of the following statements is true?

(a) f has only 1 saddle point.

(b) f has only 1 local maximum point.

(c) f has only 1 local minimum point.

(d) f has 1 saddle point and 1 local minimum.

(e) f has 1 saddle point and 1 local maximum.

(f) f has 1 local maximum and 1 local minimum.

$f_{xx} = 2$ $\quad f_{xy} = -4$ $\quad f_{yy} = 2$

critical points $\left.\begin{array}{l} 2x-4y=0 \\ 2y-4x=0 \end{array}\right\}$ $x=0,\ y=0$ $\qquad D = \begin{vmatrix} 2 & -4 \\ -4 & 2 \end{vmatrix} = 4-16<0$

7. Find the volume of the solid below the plane $4x + y + z = 6$ and above the disk $x^2 + y^2 \leq 1$.

(a) π

(b) 2π

(c) 3π

(d) 4π

(e) 5π

(f) 6π

Volume $= \iint_D 6 - 4x - y \, dA$ D: disc of radius 1

in polar coordinates volume $= \int_0^{2\pi} \int_0^1 \underbrace{r(6 - 4r\cos\theta - r\sin\theta)}_{6r - 4r^2\cos\theta - r^2\sin\theta} \, dr \, d\theta$

$= \int_0^{2\pi} 3r^2 - \frac{4r^3}{3}\cos\theta - \frac{r^3}{3}\sin\theta \Big|_{r=0}^{r=1} d\theta$

$= \int_0^{2\pi} 3 - \frac{4}{3}\cos\theta - \frac{1}{3}\sin\theta \, d\theta$

$= 3\theta - \frac{4}{3}\sin\theta + \frac{1}{3}\cos\theta \Big|_0^{2\pi} = 3(2\pi) = 6\pi$

8. Let D be the region bounded by the lines $y = x$, $y = 1$ and $x = 0$. Evaluate

$$\iint_D \frac{\sin y}{y}\, dA.$$

(a) 1

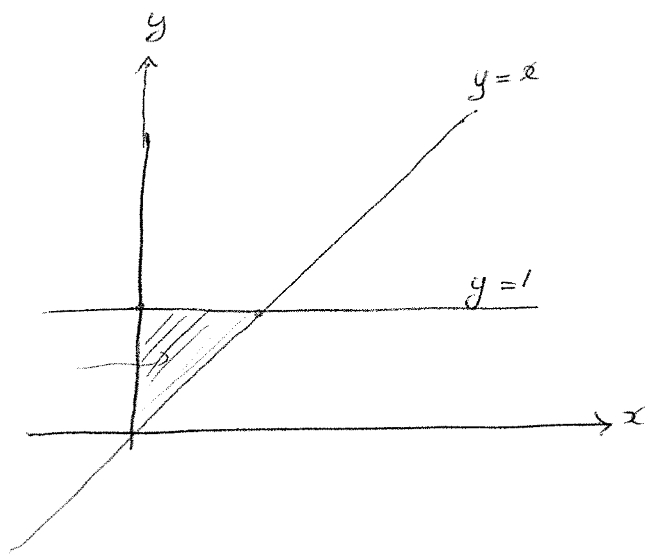
(b) $1 + \sin(1)$

(c) $\frac{\pi}{2}$

(d) $1 - \cos(1)$

(e) $-\cos(1)$

(f) $\frac{\pi}{2} - \cos(1)$



$D: \; 0 \le y \le 1 \qquad 0 \le x \le y$

$$\iint_D \frac{\sin y}{y}\, dA = \int_0^1 \int_0^y \frac{\sin y}{y}\, dx\, dy$$

$$= \int_0^1 \frac{\sin y\, x}{y} \Big|_{x=0}^{x=y} dy = \int_0^1 \sin y \; dy = -\cos y \Big|_0^1$$

$$= -\cos 1 + 1$$

9. Use the transformation $x = u^2$, $y = v^2$ to find the area of the region in the first quadrant enclosed by the curves $x = 0$, $y = 0$, and $\sqrt{x} + \sqrt{y} = 1$.

(a) $\frac{1}{2}$

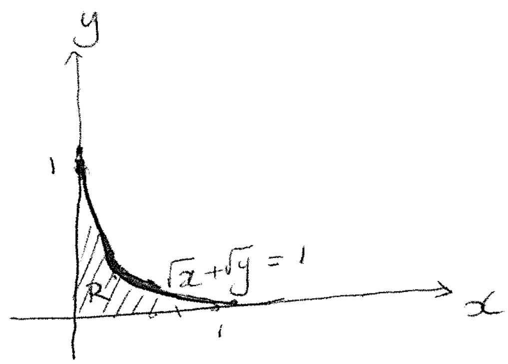
(b) $\frac{\sqrt{2}}{6}$

(c) $\frac{\sqrt{2}}{3}$

(d) $\frac{\sqrt{2}-1}{3}$

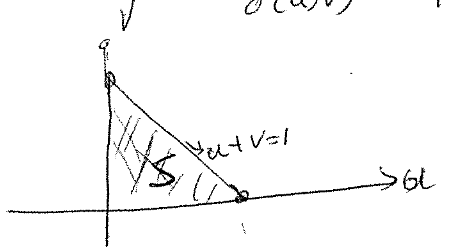
(e) $\frac{1}{6}$

(f) $\frac{1}{3}$



$x = u^2$ $\quad y = v^2$ $\quad \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2u & 0 \\ 0 & 2v \end{vmatrix} = 4uv$

$S:$ $\quad u + v \leq 1$



area of $R = \iint_R 1\, dA = \iint_S 4uv\, dA = \int_0^1 \int_0^{1-u} 4uv\, dv\, du$

$= \int_0^1 2uv^2 \Big|_{v=0}^{v=1-u} du = \int_0^1 \underbrace{2u(1-u)^2}_{2u+2u^3-4u^2} du = u^2 + \frac{2}{4}u^4 - \frac{4}{3}u^3 \Big|_0^1 = 1 + \frac{1}{2} - \frac{4}{3}$

$= \frac{1}{6}$

10. What is the work done by the force $\mathbf{F} = \langle y, y - x \rangle$ in moving a particle along the line segment from $(2,3)$ to $(1,1)$?

(a) 0

(b) 6

(c) -3

(d) 2

(e) -2

(f) $\frac{1}{2}$

$0 \le t \le 1$

$\vec{r}(t) = t\langle 1,1\rangle + (1-t)\langle 2,3\rangle$

$= \langle t, t\rangle + \langle 2-2t, 3-3t\rangle$

$= \langle \underbrace{2-t}_{x}, \underbrace{3-2t}_{y}\rangle \qquad x'(t) = -1 \qquad y'(t) = -2$

$\text{work} = \int_C \vec{F}\cdot d\vec{r} = \int_0^1 \langle 3-2t, \underbrace{3-2t-(2-t)}_{1-t}\rangle \cdot \underbrace{\langle -1,-2\rangle}_{\vec{r}\,'(t)}\, dt$

$= \int_0^1 -(3-2t) - 2(1-t)\, dt$

$= \int_0^1 -5 + 4t\, dt = -5t + \frac{4t^2}{2}\Big|_0^1 = -5 + \frac{4}{2} \cdot 2 = -3$

11. Which of the following vector fields are conservative?

$$\mathbf{F_1} = \langle 3x^2 + 2xy,\ x^2 + 1 \rangle;$$
$$\mathbf{F_2} = \langle \sin(xy),\ x \sin y \rangle;$$
$$\mathbf{F_3} = \langle xe^y,\ e^{xy} \rangle.$$

(a) Only F_1.

(b) Only F_2.

(c) Only F_3

(d) None of them is conservative.

(e) Only F_1 and F_3 are conservative.

(f) Only F_2 and F_3 are conservative.

12. Evaluate

$$\int_C 2x\cos y\,dx - x^2\sin y\,dy$$

if C is the parabola $y = (x-1)^2$ from $(1,0)$ to $(0,1)$.

(a) 0

(b) $\cos(1)$

(c) 1

(d) -1

(e) $\dfrac{\cos(1)}{2}$

(f) $\sqrt{2}$

$\vec{F} = \langle 2x\cos y, -x^2\sin y\rangle$ is conservative and $\vec{F} = \nabla\langle \underbrace{x^2\cos y}_{f}\rangle$

$\Rightarrow \int_C 2x\cos y\,dx - x^2\sin y\,dy = f(0,1) - f(1,0) = 0 - \cos 0 = -1$

13. Let C be the ellipse $9x^2 + 4y^2 = 36$ with positive orientation. If $\mathbf{F} = \langle xy,\, 1\rangle$, then evaluate

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

(a) $12\pi - 2$

(b) $6\pi + 2$

(c) 6π

(d) 4π

(e) 12π

(f) 0

$$\frac{9x^2}{36} + \frac{4y^2}{36} = 1 \Rightarrow \left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1 \Rightarrow \begin{array}{l} x = 2\cos t \\ y = 3\sin t \\ 0 \le t \le 2\pi \end{array}$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \langle 6\cos t \sin t, 1\rangle \circ \underbrace{\langle -2\sin t, 3\cos t\rangle}_{\vec{r}\,'(t)} \, dt$$

$$= \int_0^{2\pi} -12\cos t \sin^2 t + 3\cos t \, dt = -4\sin^3 t + 3\sin t \Big|_0^{2\pi} = 0$$

14. Let $f(x, y) = 4xy^2$ and let C be the part of the circle $x^2+y^2 = 1$ in the second quadrant. Evaluate

$$\int_C f(x, y)\, ds.$$

(a) 4

(b) $-\frac{4}{3}$

(c) 0

(d) -12

(e) $\frac{16}{9}$

(f) $4\sqrt{2}$

$C: \vec{r}(t) = \langle \cos t, \sin t \rangle \quad \frac{\pi}{2} \le t \le \pi \quad |\vec{r}\,'(t)| = 1$

$\int_C f(x,y)\, ds = \int_{\frac{\pi}{2}}^{\pi} 4 \cos t \sin^2 t \, dt = \frac{4}{3} \sin^3 t \Big|_{\frac{\pi}{2}}^{\pi} = -\frac{4}{3}$

15. Evaluate the line integral

$$\int_C x^2\,dx + (y^3 + 2yx)\,dy$$

where C is the triangle cut out with lines $x = 1$, $x + y = 2$, and $y = x - 2$ oriented counter clockwise.

(a) 0

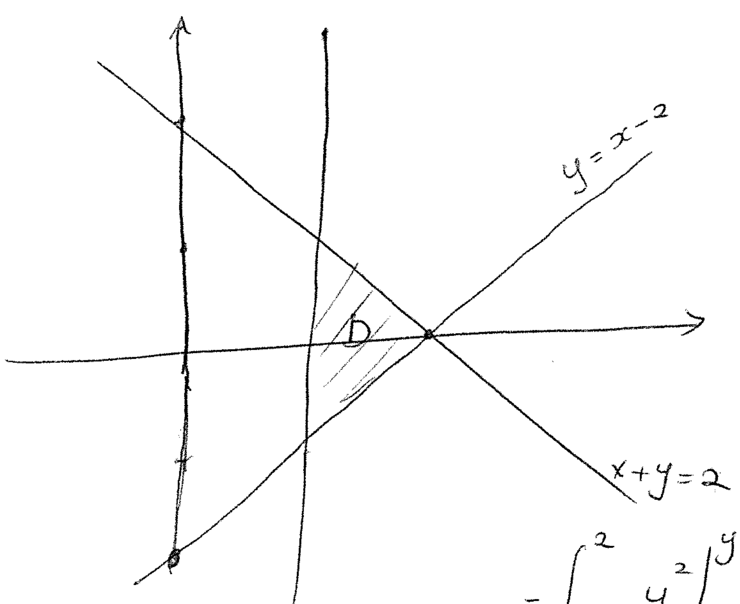
(b) 2

(c) $\frac{6}{5}$

(d) $\frac{8}{3}$

(e) -2

(f) -4



D: $1 \le x \le 2 \quad x-2 \le y \le 2-x$

By Green's theorem

$$\int_C \underbrace{x^2}_{P}\,dx + \underbrace{(y^3 + 2yx)}_{Q}\,dy$$

$$= \iint_D 2y\,dA = \int_1^2 \int_{x-2}^{2-x} 2y\,dy\,dx$$

$$= \int_1^2 y^2 \Big|_{y=x-2}^{y=2-x} dx = \int_1^2 0\,dx = 0$$

16. Suppose D is a simply connected region with an area of 1 and a smooth, positively oriented boundary curve C. If $f(x, y) = x - 2y$, then find

$$\int_C f(x, y)\, dx.$$

(a) 1

(b) 0

(c) -1

(d) $-\pi$

(e) 2

(f) -2

$$\int_C x - 2y \, dx = \int_C x\, dx - 2\int_C y\, dx \underset{\text{Green's Theorem}}{\overset{\text{by}}{=}} 0 + 2 \text{ area}$$

$$= 2$$