HIGHER-ORDER TANGENT LINES TO SMOOTH HYPERSURFACES

ROYA BEHESHTI AND SHIBASHIS MUKHOPADHYAY

ABSTRACT. We prove that for every smooth hypersurface $X \subset \mathbb{P}^n$ of degree $d \geq n \geq 5$ over an algebraically closed field of characteristic zero, the space of pointed lines (ℓ, p) having intersection multiplicity at least six with X at p has the expected dimension $2n-7$. The proof uses deformations of ruled surfaces preserving prescribed contact with X. As an application, we show that a smooth hypersurface of degree $d \geq 6$ in $\mathbb{P}^5$ contains at most $225d^3 - 1370d^2 + 1800d$ planes. This bound has optimal order of growth in d.

1. INTRODUCTION

Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree d over an algebraically closed field of characteristic 0. For $1 \leq k \leq d+1$, let $V_k(X)$ denote the subscheme of the space of pointed lines in $\mathbb{P}^n$ parametrizing pairs (ℓ, p) such that ℓ has intersection multiplicity at least k with X at p. The expected dimension of $V_k(X)$ is $2n-k-1$, and every irreducible component has dimension at least this large. We ask whether smoothness forces equality in the following range.

Question 1.1. *Fix $k \geq 1$. If X is smooth and $d \geq n \geq k-1$, does $V_k(X)$ have the expected dimension?*

When $k = d+1$, the contact condition forces $\ell \subset X$, so $V_{d+1}(X)$ is the universal family over the Fano variety of lines on X. The Debarre–de Jong conjecture predicts that for every smooth hypersurface $X \subset \mathbb{P}^n$ of degree $d \leq n$ the Fano variety of lines $F(X)$ has the expected dimension $2n-d-3$. Therefore, for each fixed degree d, an affirmative answer to the above question when $n = d$ and $k = d+1$ would imply the Debarre–de Jong conjecture for smooth hypersurfaces of degree d in $\mathbb{P}^d$, and hence for all smooth Fano hypersurfaces of degree d [3]. Lines having prescribed high-order contact and the associated contact cones were studied by Jennings [6]. For general hypersurfaces, Bastianelli, Ciliberto, Flamini, and Supino [1] obtained dimension results for these cones and applications to measures of irrationality.

The answer is affirmative when $k \leq 4$. Patel, Riedl, Smith, and Tseng [8] proved that $V_5(X)$ has the expected dimension whenever $d \geq n \geq 4$. In this paper, we prove the next case.

Theorem 1.2. *Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree d, with $d \geq n \geq 5$. Then $V_6(X)$ has the expected dimension.*

Combining the theorem with an intersection-theoretic calculation from [8] gives an explicit bound on the number of 2-planes contained in a smooth hypersurface in $\mathbb{P}^5$:

Corollary 1.3. *If $X \subset \mathbb{P}^5$ is a smooth hypersurface of degree $d \geq 6$, then the number of 2-planes contained in X is at most*

$$225d^3 - 1370d^2 + 1800d.$$

For $d \geq 3$, the number of 2-planes in a smooth hypersurface of degree d in $\mathbb{P}^5$ is finite [2]. The corollary improves the $O(d^4)$ bound of [8] to a bound with optimal order of growth: the Fermat hypersurface of degree $d \geq 3$ in $\mathbb{P}^5$ contains at least $15d^3$ planes. The use of high-contact loci to bound the number of linear spaces in smooth hypersurfaces was also discussed by Starr [12]. For comparison, Degtyarev, Itenberg, and Ottem [5] proved, using lattice theory, that a smooth complex cubic fourfold contains at most 405 planes.

The idea of the proof is inspired by [3], where it is proved that the Fano variety of lines on every smooth Fano hypersurface of degree ≤ 6 has the expected dimension. The case $d = n = 5$ of our theorem follows from this result. As we show in Lemma 4.1, the case $d = 6$ also follows directly from the known case $d = n = 6$ of the Debarre-de Jong conjecture. For $d \geq 7$, the same construction no longer reduces the problem to the Fano scheme of lines on an auxiliary hypersurface. Instead, we adapt the deformation-theoretic approach of [3] to the cones swept out by lines having intersection multiplicity at least 6 with X.

Section 2 recalls preliminary results on lines with high-order contact. In Section 3, we reduce the theorem to excluding a four-dimensional family of lines in $\mathbb{P}^5$ having contact order at least six with X and sweeping out a hypersurface Z. In Section 4, we study deformations of the cones swept out by lines of the family through a general point of Z, and complete the proof by a cohomological vanishing calculation on the associated ruled surfaces.

Acknowledgment. We are grateful to Raymond Cheng and Eric Riedl for helpful conversations. Roya Beheshti was partially supported by Simons Foundation grant 00007742.

AI disclosure: The core ideas and all proofs in this work were developed by the authors, inspired by [3]. ChatGPT was used during the preparation of the paper solely for proofreading and expository assistance. During this review, a gap in Section 3 was identified, which was subsequently resolved by the authors.

2. Preliminaries

In this section, we recall several facts about lines with high-order contact. Throughout, we work over an algebraically closed field of characteristic 0.

We write $F(X)$ for the Fano variety of lines on X. Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree $d \geq 2$, and let $p \in X$. Choose homogeneous coordinates so that $p = (1 : 0 : \cdots : 0)$, and write a defining equation of X as

$$f = x_0^{d-1} f_1 + x_0^{d-2} f_2 + \cdots + f_d,$$

where each f_i is homogeneous of degree i in $x_1, \ldots, x_n$. For $1 \leq k \leq d+1$, the scheme of lines through p having contact of order at least k with X is identified with

$$\Sigma_k^p(X) = V(f_1, \ldots, f_{k-1}) \subset \mathbb{P}^{n-1}.$$

The lines contained in X are understood to have infinite contact order. In particular, the scheme of lines contained in X and passing through p is

$$F^p(X) = V(f_1, \ldots, f_d).$$

A basic question is whether these contact equations meet properly when p is general. The first nonlinear condition occurs for contact of order at least 3. Since X is smooth, $f_1 \neq 0$, and the restriction of f_2 to the hyperplane $V(f_1)$ represents the second fundamental form

II_p of X at p. This form is nondegenerate at a general point. More generally, we have the following degeneracy bound, recalled in [8, Theorem 3.1].

Proposition 2.1. *For $0 \le i \le n-1$, let $R_i = \{p \in X : \operatorname{rank} \mathrm{II}_p \le n-1-i\}$. Then*

$$\operatorname{codim}_X R_i \ge i.$$

We next derive a consequence of Landsberg's theorem [7], whose formulation in [8, Theorem 1.1] we use below.

Proposition 2.2. *Let X be a reduced, irreducible hypersurface in $\mathbb{P}^n$. Suppose that G is an irreducible component of $F(X)$ whose lines sweep out X. Let p be a general point of X and let Σ be an irreducible component of $G^p = \{\ell \in G : p \in \ell\}$. Set $r = \dim \Sigma$ and $m = n-1-r$. Then*

$$\deg \Sigma \le m!.$$

If equality holds, then Σ is a complete intersection of type $(1, 2, \ldots, m)$ in $\mathbb{P}^{n-1}$.

Proof. Let $d = \deg X$. Choose homogeneous coordinates such that $p = (1 : 0 : \cdots : 0)$, and write a defining equation of X as

$$f = x_0^{d-1} f_1 + x_0^{d-2} f_2 + \cdots + f_d,$$

where each f_i is homogeneous of degree i in $x_1, \ldots, x_n$. Since G is an irreducible component of $F(X)$ and p is a general point of X, Σ is an irreducible component of $F^p(X) = V(f_1, \ldots, f_d)$. We claim that Σ is an irreducible component of $V(f_1, \ldots, f_m)$. This is immediate if $m = d$. If $m < d$ and the claim fails, there is an irreducible component Σ' of $V(f_1, \ldots, f_m)$ that properly contains Σ, so

$$\dim \Sigma' \ge r + 1.$$

Every line parametrized by Σ' has contact of order at least $m+1$ with X at p. The expected dimension of this space is

$$n - (m+1) = r.$$

Since p is a general point of X and since Σ' is larger than expected, by [8, Theorem 1.1], all lines parametrized by Σ' are contained in X. Thus $\Sigma' \subset F^p(X)$, contradicting the fact that Σ is an irreducible component of $F^p(X)$.

The degree bound now follows since any irreducible component of $V(f_1, \ldots, f_m)$ has degree $\le m!$, and if the degree is equal to $m!$, this component should be equal to the complete intersection $V(f_1, \ldots, f_m)$.

□

Let $V_k(X)$ denote the scheme of pointed lines (ℓ, p) with $p \in \ell$ and contact of order at least k with X at p. Patel–Riedl–Smith–Tseng prove that a component of this scheme has the expected dimension whenever its lines sweep out the ambient projective space [8, Theorem 2.8].

Theorem 2.3. *Let W be an irreducible component of $V_k(X)$, with $1 \le k \le d$. If*

$$\overline{\bigcup_{(\ell,p)\in W} \ell} = \mathbb{P}^n,$$

then

$$\dim W = 2n - k - 1.$$

The same deformation-theoretic argument gives a nonexistence statement for high-contact lines covering a subvariety of $\mathbb{P}^n$.

Proposition 2.4. *Let $Z \not\subset X$ be an irreducible subvariety of $\mathbb{P}^n$ of dimension r. There is no irreducible family F of lines sweeping out Z such that a general line parametrized by F is contained in the smooth locus of Z and has contact of order at least $r+2$ with X at a smooth point of $X \cap Z$.*

Proof. We adapt the proof of [8, Theorem 2.8]. Put $k = r+2$. Since $Z \not\subset X$ and F sweeps out Z, a general line parametrized by F is not contained in X. Its contact points with X are therefore finite. After replacing F by an irreducible generically finite cover, we may mark a contact point p of order at least k that is smooth on $X \cap Z$. We restrict to the open subset on which every line is contained in the smooth locus of Z.

Consider the family T of all parametrizations $\iota : \mathbb{P}^1 \xrightarrow{\sim} \ell$ of these marked lines satisfying $\iota(0) = p$. For a fixed $q \in \mathbb{P}^1 \setminus \{0\}$, the evaluation map $\mathrm{ev}_q : T \longrightarrow Z$ is dominant, so by generic smoothness, its differential is surjective at a general point $t \in T$.

Fix such a parametrization and identify $\mathbb{P}^1$ with ℓ. Let $kp \subset \ell$ denote the length-k subscheme supported at p. Since the defining section f of X vanishes on kp, its differential induces a well-defined map

$$T_Z|_\ell \longrightarrow \mathcal{O}_\ell(d)|_{kp}.$$

Since $X \cap Z$ is smooth at p, this map is nonzero on the fiber at p and hence is surjective. Its kernel M is a vector bundle of rank r, and

$$\deg M = \deg(T_Z|_\ell) - k.$$

We have the short exact sequence

$$0 \longrightarrow \mathcal{O}_\ell(2) \longrightarrow T_Z|_\ell \longrightarrow N_{\ell/Z} \longrightarrow 0.$$

Since $N_{\ell/Z} \hookrightarrow N_{\ell/\mathbb{P}^n} \simeq \mathcal{O}_\ell(1)^{\oplus(n-1)}$, every summand in the splitting of $N_{\ell/Z}$ has degree at most 1. Since its rank is $r-1$, we obtain

$$\deg(T_Z|_\ell) \le 2 + (r-1) = r+1, \qquad \deg M \le (r+1) - (r+2) = -1.$$

Every tangent vector to T at t induces a global section of $T_Z|_\ell$. Since the family preserves contact of order at least k at the fixed source point 0, this section belongs to $H^0(\ell, M)$. Surjectivity of the differential of ev_q therefore gives r such sections whose values at q are linearly independent. Their exterior product is a nonzero global section of $\det M$, contradicting $\deg M < 0$. □

We will also use the following classical classification of projective varieties containing large families of lines.

Proposition 2.5 (Segre's classification; see [10, Theorem 1])**.** *Let $Y \subset \mathbb{P}^N$ be an integral projective variety of dimension $m \ge 3$ over an algebraically closed field of characteristic zero. Suppose that*

$$\dim F(Y) \ge 2m - 3.$$

Then Y is one of the following:

1. *a linear space;*
2. *a possibly singular quadric hypersurface in its linear span;*
3. *a scroll swept out by a one-parameter family of $(m-1)$-planes.*

3. Geometric reductions

In this section, we use the results of the previous section to rule out some of the simpler cases in the proof of the main theorem. By the following lemma, it is enough to consider the case $n = 5$.

Lemma 3.1. *Fix $k \geq 3$ and $d \geq k-1$. Suppose that Question 1.1 has an affirmative answer for $n = k-1$. Then it has an affirmative answer for every n with $d \geq n \geq k-1$.*

Proof. We prove the statement by induction on n. When $n = k-1$, by our hypothesis, V_k has the expected dimension. Suppose that $n > k-1$, and let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree $d \geq n$. Assume that $W \subset V_k(X)$ is an irreducible component with $\dim W > 2n-k-1$. Let F be its image in the Grassmannian of lines.

Since $\dim W > 2n-k-1 \geq n-1$, the lines parametrized by F do not all pass through the same point. So by [3, Lemma 4.4], for a general hyperplane $H \subset \mathbb{P}^n$, the locus

$$F_H = \{[\ell] \in F : \ell \subset H\}$$

is nonempty and has codimension 2 in F. Hence if $W_H = \{(\ell, p) \in W : \ell \subset H\}$, then $W_H \subset V_k(X \cap H)$, and

$$\dim W_H = \dim W - 2 > 2(n-1) - k - 1,$$

so $\dim V_k(X \cap H)$ is larger than expected. Since H is general, $X \cap H$ is non-singular, contradicting the induction hypothesis. □

Applying the lemma with $k = 6$, we may restrict to $n = 5$. If $d = 5$, every line having contact of order at least 6 with X is contained in X. Since $\dim F(X) = 2$ by [3], we have $\dim V_6(X) = 3$, as required. We may therefore assume that $d \geq 6$.

Suppose, to the contrary, that $\dim V_6(X) \geq 4$. Let W be an irreducible component of $V_6(X)$ of dimension greater than expected, let F be its image in the Grassmannian of lines, and let $Z \subset \mathbb{P}^5$ be the subvariety swept out by these lines.

Since $\deg X \geq 5$, X is not swept out by lines. We claim that $\dim F(X) \leq 2$. Suppose that G is an irreducible component of $F(X)$ with $\dim G \geq 3$. Since a surface contains at most a two-dimensional family of lines, the lines parametrized by G sweep out a divisor $Y \subset X$. By Proposition 2.5, Y is either a linear space, a quadric, or a scroll swept out by a one-parameter family of 2-planes. The first two possibilities are excluded because every nonzero effective divisor on X has degree at least $\deg X$ by the Lefschetz theorem. The scroll case is also impossible, since a smooth hypersurface of degree > 2 in $\mathbb{P}^5$ contains only finitely many 2-planes [12].

The locus of pointed lines contained in X therefore has dimension at most 3, so a general line parametrized by W is not contained in X. Therefore,

$$\dim F = \dim W \geq 4, \qquad Z \not\subset X.$$

By [8, Theorem 2.8], we also have $Z \neq \mathbb{P}^5$. Thus $\dim Z$ is either 3 or 4.

We will rule out the following cases:

(1) $\dim Z = 3$;
(2) $\dim Z = 4$ and F is contained in an irreducible component of $F(Z)$ of dimension at least 5.

Case 1. Suppose first that $\dim Z = 3$. Since Z contains a four-dimensional family of lines, it must be a 3-plane, and F is its entire family of lines. Set

$$S = X \cap Z.$$

The singular locus of S has dimension at most 1: under the finite Gauss map of X, it maps into the pencil of hyperplanes containing Z. The lines in Z meeting $\operatorname{Sing} S$ therefore form a family of dimension at most 3. Hence a general line of F avoids $\operatorname{Sing} S$, contradicting Proposition 2.4.

We may therefore assume that Z is a hypersurface in $\mathbb{P}^5$.

Case 2. Suppose now that $\dim Z = 4$ and F is contained in an irreducible component of $F(Z)$ of dimension ≥ 5. Since $\dim F(Z) \geq 5$, by Proposition 2.5, Z is either a linear space, a quadric, or a scroll. We get a contradiction in each case:

(1) Z is a linear subspace. In this case, $Z \simeq \mathbb{P}^4$. This possibility is ruled out by Proposition 2.4 since the intersection $Z \cap X$ has only finitely many singular points, and through every such point there is a 3-dimensional family of lines in Z.

(2) Z is an irreducible quadric hypersurface. The singular locus of Z is either empty, a point, a line, or a plane. If the singular locus of Z is empty or a point, then the family of lines on Z meeting the singular locus of Z has dimension at most 3. Moreover, $Z \cap X$ has at most finitely many singular points (see for example [4, Lemma 2.1]). Therefore, not every line parametrized by F can meet the singular locus of Z or the singular locus of $Z \cap X$. Proposition 2.4 therefore gives a contradiction.

If the singular locus of Z is a line or a plane, then Z contains a 1-dimensional family of 3-planes, i.e. Z is a scroll. This is ruled out in the next case.

(3) Suppose that Z is a scroll ruled by a one-parameter family of 3-planes. Let G be an irreducible component of $F(Z)$ of dimension at least 5 containing F. Then by [8, Proposition 3.7], there is a ruling $\{\Lambda_t \simeq \mathbb{P}^3\}_{t \in T}$ of Z such that G parametrizes the lines contained in its members. In particular, every line parametrized by F lies in a ruling 3-plane.

For general t, a component F_t of the family of lines of F in Λ_t has dimension at least 3 and sweeps out Λ_t. By Proposition 2.4, a general marked contact point of this family lies in $\operatorname{Sing}(X \cap \Lambda_t)$. This singular locus has dimension at most 1, so the only possibility is that $\dim \operatorname{Sing}(X \cap \Lambda_t) = 1$, and there is a 1-dimensional irreducible component C_t of the singular locus such that for every point $p \in C_t$, every line in Λ_t through p has contact order at least six with X at p. The next lemma shows this is not possible.

Lemma 3.2. *Let $X = V(f) \subset \mathbb{P}^n$ be a smooth hypersurface of degree $d > 2$. There are at most finitely many codimension-2 linear subspaces $\Lambda \subset \mathbb{P}^n$ admitting a curve C in $\Lambda \cap X$ such that $f|_\Lambda$ has multiplicity ≥ 3 at every point of C.*

Proof. Suppose to the contrary that there is a nonconstant one-parameter family of such $(n-2)$-planes. After a finite base change and shrinking the base, we may assume that the family is parametrized by a smooth curve T and that there is a flat family of integral projective curves $\mathcal{C} \to T$ such that $C_t \subset \Lambda_t$ and $f|_{\Lambda_t}$ vanishes to order at least three at a general point of C_t.

Fix a general $t \in T$, and write $\Lambda = \Lambda_t$ and $C = C_t$. Let

$$\nu : \widetilde{C} \longrightarrow C$$

be the normalization, and let $\varphi : \widetilde{C} \to \mathbb{P}^n$ be the composition of ν with the inclusion of C. Set $L = \varphi^* \mathcal{O}_{\mathbb{P}^n}(1)$.

Choose coordinates so that $\Lambda = V(x_0, x_1)$, and set

$$G_a = \left.\frac{\partial f}{\partial x_a}\right|_{\Lambda}, \qquad a = 0, 1.$$

Since X is smooth along C and C is in the singular locus of $X \cap \Lambda$, the map

$$\beta : \varphi^* \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus 2} \longrightarrow \varphi^* \mathcal{O}_{\mathbb{P}^n}(d),$$

given by the pullbacks of G_0 and G_1, is surjective. Its kernel K is a line bundle with $\deg K = (2-d) \deg L < 0$, hence $H^0(\widetilde{C}, K) = 0$.

The family Λ_t determines a nonzero section

$$\alpha = (\ell_0, \ell_1) \in H^0(\Lambda, N_{\Lambda/\mathbb{P}^n}) \simeq H^0(\Lambda, \mathcal{O}_\Lambda(1))^{\oplus 2}.$$

Set

$$\sigma = \ell_0 G_0 + \ell_1 G_1 \in H^0(\Lambda, \mathcal{O}_\Lambda(d)).$$

We claim that σ vanishes to order at least two at a general point of C. Let R be the local coordinate ring of Λ and $I \subset R$ the ideal of C near a general point where $\mathcal{C} \to T$ is smooth. Set

$$R_\epsilon = R[\epsilon]/(\epsilon^2),$$

and let $J \subset R_\epsilon$ be the ideal of the first-order deformation of C, so that $J \bmod \epsilon = I$. Trivialize $\mathcal{O}_\Lambda(d)$ locally, let $g \in R$ represent $f|_\Lambda$, and let $g_\epsilon \in R_\epsilon$ be the local equation obtained by restricting f to the first-order deformation Λ_ϵ, so

$$g_\epsilon = g + \epsilon\sigma.$$

The contact hypothesis gives $g_\epsilon \in J^3$. Since $J \subset IR_\epsilon + (\epsilon)$, we have

$$J^3 \subset (IR_\epsilon + (\epsilon))^3 = I^3 R_\epsilon + \epsilon I^2 R_\epsilon.$$

Comparing coefficients of ϵ gives $\sigma \in I^2$, proving the claim.

Therefore, σ and all its first partial derivatives on Λ vanish at general points of C, and hence on all of C. In particular, $\beta\big(\nu^*(\alpha|_C)\big) = 0$. Thus $\nu^*(\alpha|_C)$ is a global section of K, so it must be zero since $\deg K < 0$. It follows that

$$\ell_0|_C = \ell_1|_C = 0.$$

We now apply the same negative-degree argument to the coefficients of ℓ_0 and ℓ_1. Taking derivative, we get that for every $2 \le i \le n$,

$$0 = \left.\frac{\partial \sigma}{\partial x_i}\right|_C = \frac{\partial \ell_0}{\partial x_i} G_0|_C + \frac{\partial \ell_1}{\partial x_i} G_1|_C.$$

Tensoring the sequence defining K by L^{-1} gives

$$0 \longrightarrow K \otimes L^{-1} \longrightarrow \mathcal{O}_{\widetilde{C}}^{\oplus 2} \xrightarrow{\beta \otimes L^{-1}} L^{\otimes(d-1)} \longrightarrow 0.$$

The above identity shows that $(\frac{\partial \ell_0}{\partial x_i}, \frac{\partial \ell_1}{\partial x_i})$ defines a global section of $K \otimes L^{-1}$. But

$$\deg(K \otimes L^{-1}) = (1-d)\deg L < 0,$$

so this section is zero. Therefore all coefficients of the linear forms ℓ_0 and ℓ_1 are zero. This contradicts $\alpha \neq 0$.

□

Summarizing, we obtain the following proposition.

Proposition 3.3 (Reduction to the remaining case)**.** *Let $W \subset V_6(X)$ be an irreducible component of dimension greater than expected, let F be its image in the Grassmannian of lines, and let $Z \subset \mathbb{P}^5$ be the subvariety swept out by these lines. Then $\dim Z = 4$, and F is a four-dimensional irreducible component of $F(Z)$.*

Let $p \in Z$ be a general point, and let Σ be an irreducible component of

$$F^p = \{\ell \in F : p \in \ell\}.$$

Then Σ is a curve in $\mathbb{P}(T_pZ) \simeq \mathbb{P}^3$, and Proposition 2.2 gives

$$\deg \Sigma \leq 6.$$

If equality holds, then Σ is a complete intersection of type $(2,3)$ in $\mathbb{P}(T_pZ)$.

4. Proof of the main theorem

In this section we prove the main theorem. We first treat the case $d = 6$ using the main result of [3].

Lemma 4.1. *Theorem 1.2 holds for $d = 6$.*

Proof. Let $X = V(f)$, and let $\tilde{X} \subset \mathbb{P}^6$ be the hypersurface defined by $x_6^6 = f$. Then $\tilde{X}$ is non-singular. Projection from the point $(0 : \cdots : 0 : 1)$ gives a morphism $\pi : \tilde{X} \to \mathbb{P}^5$ branched along X. If a line ℓ in $\mathbb{P}^5$ has intersection multiplicity 6 at a point with X, then $\pi^{-1}(\ell)$ is the union of 6 lines in $\tilde{X}$. By [3] $\dim F(\tilde{X}) = 3$, so $\dim V_6(X) = 3$. □

From now on we assume $d \geq 7$. By Proposition 3.3, it remains to rule out the following situation: $X \subset \mathbb{P}^5$ is a smooth hypersurface of degree $d \geq 7$, and $W \subset V_6(X)$ is an irreducible 4-dimensional family whose lines sweep out a hypersurface $Z \subset \mathbb{P}^5$. We also assume that the image of W in the Grassmannian of lines is an irreducible component G of $F(Z)$.

Since X is not covered by lines, $Z \neq X$. Let q be a general point of Z. Then $q \notin X$ and there is a one-dimensional family of lines through q having contact order at least 6 with X. Choose an irreducible component of this family, and let S_0 be the surface swept out by its lines.

The surface S_0 is a cone with vertex q over an irreducible curve B_0, obtained by intersecting S_0 with a general hyperplane not containing q. Let $\nu : B \to B_0$ be the normalization, and set

$$\mathcal{N} = \nu^* \mathcal{O}_{B_0}(-1).$$

Normalizing the cone and resolving its vertex gives the smooth ruled surface

$$\pi : S = \mathbb{P}_B(\mathcal{O}_B \oplus \mathcal{N}) \longrightarrow B$$

and a morphism

$$\varphi : S \longrightarrow \mathbb{P}^5.$$

The morphism φ is birational onto S_0 and maps each ruling fiber isomorphically to a line through q having contact order at least 6 with X at a marked contact point given by W. The contact incidence gives a (not necessarily smooth) multisection D of π.

We next discuss deformations of morphisms from smooth surfaces to $\mathbb{P}^n$ that preserve a prescribed order of contact with a hypersurface X along a divisor. We will apply this discussion to the restriction of φ to the smooth surface $S \setminus \mathrm{Sing}(S)$ and the non-singular divisor $D \setminus \mathrm{Sing}(D)$ obtained above. The relation between logarithmic tangent sheaves and deformations preserving the inverse image of a divisor is discussed in [9, § 2.3].

4.1. **Deformations keeping contact order along a divisor.** Let $\varphi : S \longrightarrow \mathbb{P}^5$ be a generically injective morphism from a smooth surface with $\varphi(S) \not\subset X$. Suppose that $r \geq 1$ and $D \subset S$ is a a smooth effective integral Cartier divisor such that $\varphi^* X$ has order of vanishing r at a general point of D. Let $\mathcal{L} = \varphi^* \mathcal{O}_{\mathbb{P}^5}(1)$. Consider the Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}^5} \to \mathcal{O}_{\mathbb{P}^5}(1)^{\oplus 6} \to T_{\mathbb{P}^5} \to 0.$$

Let $X = V(f)$. Since X is smooth, there is a surjective map $\mathcal{O}_{\mathbb{P}^5}(1)^{\oplus 6} \to \mathcal{O}_{\mathbb{P}^5}(d)$ given by the partial derivatives of f. The composition

$$\mathcal{O}_{\mathbb{P}^5} \to \mathcal{O}_{\mathbb{P}^5}(1)^{\oplus 6} \to \mathcal{O}_{\mathbb{P}^5}(d)$$

is multiplication by $d\, f$, and the composition of the pullback of these maps to S is multiplication by $d\, \varphi^* f$. Since $\varphi^* f|_{rD} = 0$, we get a surjection

$$\rho : \varphi^* T_{\mathbb{P}^5} \to \mathcal{L}^{\otimes d}|_{rD}.$$

Denote the kernel of this map by M'. Then there is an exact sequence

$$0 \longrightarrow M' \longrightarrow \varphi^* T_{\mathbb{P}^5} \xrightarrow{\rho} \mathcal{L}^{\otimes d}|_{rD} \longrightarrow 0, \tag{1}$$

and M' is locally free of rank 5.

Let N_φ denote the normal sheaf of φ, namely the cokernel of the injective map $d\varphi : T_S \longrightarrow \varphi^* T_{\mathbb{P}^5}$. Let T' and N' be the kernel and image of the map $M' \longrightarrow N_\varphi$, respectively, so we have a short exact sequence

$$0 \to T' \to M' \to N' \to 0.$$

Denote by Q the cokernel of the map

$$\rho \circ d\varphi : T_S \longrightarrow \mathcal{L}^{\otimes d}|_{rD}.$$

Then there is also an exact sequence

$$0 \longrightarrow N' \longrightarrow N_\varphi \longrightarrow Q \longrightarrow 0.$$

Lemma 4.2. *We have*

$$T' = T_S(-\log D).$$

Proof. Let I_D be the ideal sheaf of D in S. Since $\varphi^* X \geq rD$, we have $\varphi^*(f) = s_D^r \alpha$ where s_D is the canonical section of $\mathcal{O}_S(D)$ and $\alpha \in H^0(S, \mathcal{L}^{\otimes d}(-rD))$. Since we assume that the intersection multiplicity with X at a general point of D is exactly r, we have $\alpha|_D \neq 0$.

Consider the composition map

$$\delta = \rho \circ d\varphi : T_S \longrightarrow \mathcal{L}^{\otimes d}|_{rD}.$$

Working locally, let $z = 0$ be an equation for D and write

$$\varphi^* f = z^r \alpha.$$

For a local section v of T_S, we have

$$\delta(v) = v(z^r\alpha) \bmod z^r = rz^{r-1}\alpha\, v(z) \bmod z^r.$$

So the image of δ lies in

$$\mathcal{L}^{\otimes d} \otimes \frac{I_D^{r-1}}{I_D^r} \simeq \mathcal{L}^{\otimes d}(-(r-1)D)|_D \hookrightarrow \mathcal{L}^{\otimes d}|_{rD}.$$

So δ factors as

$$T_S \xrightarrow{\nu} N_{D/S} \xrightarrow{r\alpha|_D} \mathcal{L}^{\otimes d}(-(r-1)D)|_D \hookrightarrow \mathcal{L}^{\otimes d}|_{rD},$$

where

$$\nu : T_S \longrightarrow N_{D/S} \simeq \mathcal{O}_D(D)$$

is the normal map associated with the divisor D.

By definition, the kernel of ν is the logarithmic tangent bundle $T_S(-\log D)$. Since $\alpha|_D$ is nonzero, multiplication by $r\alpha|_D$ is injective. Therefore

$$\ker(\delta) = \ker(\nu) = T_S(-\log D),$$

and the assertion follows. □

Putting everything together, we get the following commutative diagram:

(2)

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T' \simeq T_S(-\log D) & \longrightarrow & T_S & \longrightarrow & N_{D/S} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle r\alpha} & & \\
0 & \longrightarrow & M' & \longrightarrow & \varphi^* T_{\mathbb{P}^5} & \longrightarrow & \mathcal{L}^{\otimes d}|_{rD} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N' & \longrightarrow & N_\varphi & \longrightarrow & Q & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Proposition 4.3. *Suppose that D is smooth and integral and that the general multiplicity of φ^*X along D is r. There is a natural injective map from the space of first-order deformations of (S, D, φ) satisfying $\varphi_\epsilon^* X \geq rD_\epsilon$ to $H^0(S, N')$.*

Proof. Let $(S_\epsilon, D_\epsilon, \varphi_\epsilon)$ be such a deformation over $k[\epsilon]/(\epsilon^2)$. Since (S, D) is a smooth pair, its deformation is locally trivial (see [11, §3.4.4]). Choose local trivializations on an open cover $\{U_i\}$ of S. On each U_i, the first-order deformations of φ are given by sections

$$v_i \in H^0(U_i, \varphi^* T_{\mathbb{P}^5}).$$

The contact condition gives $\rho(v_i) = 0$, so $v_i \in H^0(U_i, M')$. On overlaps, the infinitesimal transition maps are given by

$$\theta_{ij} \in H^0(U_i \cap U_j, T_S(-\log D)),$$

and

$$v_i - v_j = d\varphi(\theta_{ij}).$$

Thus the images of the v_i in N' glue to a global section. The injectivity is clear since the space of deformations of (S, φ) maps injectively to $H^0(S, N_\varphi)$.

□

4.2. **A family of cone surfaces.** We now construct the required family of cone surfaces. Let G be the image of W in the Grassmannian of lines, and let $e : \mathcal{I}_G \longrightarrow Z$ be the evaluation morphism from the universal family of lines over G. Since $\dim G = 4$, we have $\dim \mathcal{I}_G = 5$. The morphism e is dominant, so a general fiber of e has dimension one.

After a finite extension and possibly shrinking, we get a generically finite dominant map

$$q : T \longrightarrow Z,$$

and a smooth projective family of curves $\mathcal{B} \to T$ parametrizing lines through $q(t)$. We may assume that T is smooth and irreducible.

Pulling back the universal line over G gives a smooth family of ruled surfaces

$$\mathcal{S} \longrightarrow T$$

and an evaluation morphism

$$\Phi : \mathcal{S} \longrightarrow Z \subset \mathbb{P}^5.$$

For a general $t \in T$, write S_t for the corresponding surface and φ_t for the restriction of Φ. The incidence construction gives a section C_t of $S_t \to B_t$ contracted to $q(t)$, and φ_t is birational onto the cone swept out by the corresponding lines.

For a general $t \in T$, $q(t) \notin X$, so the contact incidence gives an integral multisection $D_t \subset S_t$ disjoint from C_t. Let $r \geq 6$ be the multiplicity of $\varphi_t^* X$ along this multisection for a general t. After a further finite extension and shrinking T, we can assume there is a flat Cartier divisor $\mathcal{D} \subset \mathcal{S}$ such that $\Phi^* X \geq r\mathcal{D}$. We may also assume the multiplicity along a general point of D_t is exactly r.

For a general member (S, D, φ), define M' using the surjection

$$\rho : \varphi^* T_{\mathbb{P}^5} \longrightarrow \mathcal{L}^{\otimes d}|_{rD},$$

and let N' be its image in N_φ, as above. The sheaf $T_S(-\log D)$ is locally free of rank 2 and

$$\det T_S(-\log D) = \omega_S^{-1}(-D).$$

Proposition 4.4. *For a general member of the family constructed above,*

$$H^0(S, \bigwedge^4 M' \otimes \omega_S(D)) \neq 0.$$

Proof. The evaluation morphism $\Phi : \mathcal{S} \to Z$ is dominant by our assumption. Since we work in characteristic zero, the differential of Φ has rank 4 at a general point of $\mathcal{S}$. Fix a general point (t, p) in $\mathcal{S}$, and write (S, D, φ) for the corresponding member. The image of

$$T_{\mathcal{S}}|_{(t,p)} \longrightarrow \varphi^* T_Z|_p \subset \varphi^* T_{\mathbb{P}^5}|_p$$

is therefore 4-dimensional. Since the kernel of

$$\varphi^* T_{\mathbb{P}^5}|_p \longrightarrow N_\varphi|_p$$

is the two-dimensional subspace $T_S|_p$, the composite

$$T_{\mathcal{S}}|_{(t,p)} \longrightarrow \varphi^* T_Z|_p \longrightarrow N_\varphi|_p$$

has image of dimension at least 2.

On the other hand, if we set $U = S \setminus \mathrm{Sing}(D)$, then the construction in Proposition 4.3 gives a linear map

$$T_t T \longrightarrow H^0(U, N'|_U).$$

So we get a commutative diagram

$$\begin{array}{ccccc} T_{\mathcal{S}}|_{(t,p)} & & \longrightarrow & & N_\varphi|_p \\ \downarrow & & & & \uparrow \\ T_t T & \longrightarrow & H^0(U, N'|_U) & \longrightarrow & N'|_p \end{array}$$

Since the image of the top row is at least 2-dimensional, the image of the map $H^0(U, N'|_U) \to N'|_p$ is at least 2-dimensional. Hence there are sections

$$\sigma_1, \sigma_2 \in H^0(U, N'|_U)$$

whose values are linearly independent at a general point. Their exterior product, $\sigma_1 \wedge \sigma_2$, is therefore a generically nonzero section of $\bigwedge^2(N'|_U)$. The short exact sequence

$$0 \longrightarrow T_S(-\log D)|_U \longrightarrow M'|_U \longrightarrow N'|_U \to 0$$

induces a map

$$\det(T_S(-\log D)|_U) \otimes \bigwedge^2(N'|_U) \longrightarrow \bigwedge^4(M'|_U).$$

This map is generically injective, and M' is locally free, therefore we get an injective map

$$\frac{\bigwedge^2(N'|_U)}{\mathrm{tors}} \hookrightarrow \bigwedge^4(M'|_U) \otimes \det(T_S(-\log D)|_U)^\vee \simeq (\bigwedge^4 M' \otimes \omega_S(D))|_U.$$

Since $\sigma_1 \wedge \sigma_2$ is generically nonzero, its image gives a nonzero section of

$$(\bigwedge^4 M' \otimes \omega_S(D))|_U.$$

This bundle is locally free, and $S \setminus U$ is finite. The section therefore extends to all of S, proving the assertion.

□

To get a contradiction and complete the proof of Theorem 1.2, it remains to prove the vanishing

$$H^0(S, \bigwedge^4 M' \otimes \omega_S(D)) = 0$$

for the family constructed above when $r \geq 6$, which we do in the following section.

4.3. **The vanishing calculation.** Let F be a fiber of $\pi : S \to B$. Set $m = D \cdot F$ and $e = \deg B_0$. Since $\varphi^* X \geq rD$, we have

$$rm \leq d. \tag{3}$$

Since φ contracts C, we have $\mathcal{L}|_C \simeq \mathcal{O}_C$. The cone construction gives $N_{C/S} \simeq \mathcal{N}$ and $C^2 = \deg \mathcal{N} = -e$.

Because $\mathcal{L}^{\otimes m}$ and $\mathcal{O}_S(D)$ have the same degree on every fiber of π,

$$\mathcal{O}_S(D) \otimes \mathcal{L}^{\otimes(-m)} \simeq \pi^* E$$

for some line bundle E on B. Restricting to C, since $C \cap D = \emptyset$ and $\mathcal{L}|_C \simeq \mathcal{O}_C$, we see that $\pi^* E|_C \simeq \mathcal{O}_C$, so

$$\mathcal{O}_S(D) \simeq \mathcal{L}^{\otimes m}.$$

Similarly, $\mathcal{L} \otimes \mathcal{O}_S(-C)$ has degree zero on every fiber, so it is the pullback of a line bundle on B Restricting to C and using the fact that $\mathcal{L}|_C \simeq \mathcal{O}_C$ gives

$$\mathcal{L} \simeq \pi^* \mathcal{N}^{-1} \otimes \mathcal{O}_S(C).$$

We now prove a handful of useful lemmas, before proving the main vanishing theorem in Proposition 4.7

Lemma 4.5. *We have*

$$\pi_* \mathcal{O}_{rD} \simeq \bigoplus_{i=0}^{rm-1} \mathcal{N}^{\otimes i}.$$

Proof. Put $N = rm$. Twisting the ideal-sheaf sequence of rD by $\mathcal{O}_S((N-1)C)$ gives

$$0 \longrightarrow \mathcal{O}_S((N-1)C - rD) \longrightarrow \mathcal{O}_S((N-1)C) \longrightarrow \mathcal{O}_{rD}((N-1)C) \longrightarrow 0.$$

The left-hand term restricts to $\mathcal{O}_{\mathbb{P}^1}((N-1)-rm) = \mathcal{O}_{\mathbb{P}^1}(-1)$ on every ruling fiber. Therefore

$$\pi_* \mathcal{O}_S((N-1)C - rD) = R^1 \pi_* \mathcal{O}_S((N-1)C - rD) = 0.$$

Since $C \cap D = \emptyset$, we have

$$\mathcal{O}_{rD}((N-1)C) \simeq \mathcal{O}_{rD}.$$

So pushing forward the above short exact sequence to B we get

$$\pi_* \mathcal{O}_{rD} \simeq \pi_* \mathcal{O}_S((N-1)C) \simeq \mathrm{Sym}^{N-1}(\mathcal{O}_B \oplus \mathcal{N}) \simeq \bigoplus_{i=0}^{N-1} \mathcal{N}^i.$$

□

Recall that B_0 is the base curve of the cone surface S_0, and $\nu : B \to B_0$ is its normalization.

Lemma 4.6. *If B and B_0 are as above, then*

$$\deg B_0 \geq 2g_B - 2$$

and the equality holds if and only if B is a smooth complete intersection of type $(1,1,2,3)$ in $\mathbb{P}^5$.

Proof. We have shown before that B_0 is an irreducible component of a complete intersection of a quadric and a cubic in $\mathbb{P}^3$. So

$$\deg B_0 \leq 6.$$

If $\deg B_0 \geq 4$, then B_0 is non-degenerate, so by Castelnuovo's bound

$$g_B \leq \frac{(\deg B_0 - 2)^2}{4}$$

which gives the desired inequality. If B_0 has degree ≤ 3, then $g_B \leq 1$, so the inequality holds.

Equality is possible only when $e = 6$ and $g_B = 4$, and in this case, B_0 is the entire complete intersection and is non-singular. □

Proposition 4.7. *If $d \geq 7$ and $r \geq 6$, then for the family constructed above,*

$$H^0(S, \bigwedge^4 M' \otimes \omega_S(D)) = 0.$$

Proof. The defining sequence for M' gives

$$\det M' \simeq \mathcal{L}^{\otimes 6}(-rD).$$

Thus Serre duality yields

$$\begin{aligned} H^0\big(S, \wedge^4 M'(K_S + D)\big) &\simeq H^0\big(S, (M')^\vee \otimes \mathcal{L}^{\otimes 6}(K_S - (r-1)D)\big) \\ &\simeq H^2\big(S, M' \otimes \mathcal{L}^{\otimes(-6)}((r-1)D)\big)^\vee. \end{aligned}$$

So it is enough to prove that the last H^2 vanishes.

Tensoring the defining sequence for M' by $\mathcal{L}^{\otimes(-6)}((r-1)D)$ gives the following short exact sequence

$$0 \to M' \otimes \mathcal{L}^{\otimes(-6)}((r-1)D) \to \varphi^* T_{\mathbb{P}^5} \otimes \mathcal{L}^{\otimes(-6)}((r-1)D) \to \mathcal{L}^{\otimes(d-6)}((r-1)D)|_{rD} \to 0.$$

Applying the long exact sequence of cohomology, it is enough to show

(1) $H^2(S, \varphi^* T_{\mathbb{P}^5} \otimes \mathcal{L}^{\otimes(-6)}((r-1)D)) = 0$

(2) The map

$$H^1(S, \varphi^* T_{\mathbb{P}^5} \otimes \mathcal{L}^{\otimes(-6)}((r-1)D)) \to H^1(S, \mathcal{L}^{\otimes(d-6)}((r-1)D)|_{rD})$$

is surjective.

Since $r \geq 6$ and $m \geq 1$, we have $(r-1)m - 5 \geq 0$, so the Euler sequence shows the first vanishing holds. So it remains to prove (2). Set $t = d + (r-1)m - 6$. We have

$$\mathcal{L}^{\otimes(d-6)}((r-1)D)|_{rD} \simeq \mathcal{L}^{\otimes t}|_{rD} \simeq \pi^* \mathcal{N}^{\otimes(-t)}|_{rD},$$

so

$$\begin{aligned} H^1(S, \mathcal{L}^{\otimes(d-6)}((r-1)D)|_{rD}) &= H^1(B, \pi_*(\pi^* \mathcal{N}^{\otimes(-t)} \otimes \mathcal{O}_{rD})) \\ &= H^1(B, \mathcal{N}^{\otimes(-t)} \otimes \pi_* \mathcal{O}_{rD}). \end{aligned}$$

By Lemma 4.5, we have

$$\pi_* \mathcal{O}_{rD} = \bigoplus_{i=0}^{rm-1} \mathcal{N}^{\otimes i}$$

so

$$H^1(S, \mathcal{L}^{\otimes(d-6)}((r-1)D)|_{rD}) = H^1(B, \bigoplus_{i=0}^{rm-1} \mathcal{N}^{\otimes(i-t)}).$$

The largest exponent is $m+5-d$, and $\deg \mathcal{N} < 0$. Since $rm \le d$, $r \ge 6$, and $d \ge 7$, we have

$$d - m - 5 \ge 1.$$

If

$$(d-m-5)e > 2g_B - 2,$$

then every summand has vanishing H^1, and the required surjectivity is automatic.

It remains to consider the case

$$(d-m-5)e \le 2g_B - 2.$$

Since $d-m-5 \ge 1$, by Lemma 4.6, if $(d-m-5)e \le 2g_B-2$, then B_0 is a smooth complete intersection of type $(1,1,2,3)$ in $\mathbb{P}^5$, and $d-m-5=1$. Since $d \ge 7$ by our assumption and $mr \le d$, this implies $d=7$, $m=1$, and $r \in \{6,7\}$. We will prove (2) in these two cases.

Since $m=1$, $\pi|_D : D \to B$ is an isomorphism, so D is a smooth section of π. Since B is a smooth complete intersection of a cubic and a quadric in $\mathbb{P}^3$, $K_B \simeq \mathcal{N}^{-1}$, and $\mathcal{L} \simeq \mathcal{O}_S(D)$. The required surjectivity is therefore

$$H^1\big(S, \varphi^* T_{\mathbb{P}^5} \otimes \mathcal{L}^{\otimes(r-7)}\big) \longrightarrow H^1(S, \mathcal{L}^{\otimes r}|_{rD}).$$

Since D is smooth, the diagram (2) is valid on all of S. Tensoring its first two rows by $\mathcal{L}^{\otimes(r-7)}$ gives the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T_S(-\log D) \otimes \mathcal{L}^{\otimes(r-7)} & \longrightarrow & T_S \otimes \mathcal{L}^{\otimes(r-7)} & \longrightarrow & N_{D/S} \otimes \mathcal{L}^{\otimes(r-7)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M' \otimes \mathcal{L}^{\otimes(r-7)} & \longrightarrow & \varphi^* T_{\mathbb{P}^5} \otimes \mathcal{L}^{\otimes(r-7)} & \longrightarrow & \mathcal{L}^{\otimes r}|_{rD} & \longrightarrow & 0
\end{array}$$

We first prove that

$$H^2(S, T_S(-\log D) \otimes \mathcal{L}^{\otimes(r-7)}) = 0.$$

Let F be a fiber of π and $p = D \cap F$. Since $T_S|_F \simeq \mathcal{O}_F \oplus \mathcal{O}_F(2)$, and since there is a short exact sequence

$$0 \longrightarrow T_S(-\log D)|_F \longrightarrow T_S|_F \longrightarrow k(p) \longrightarrow 0$$

we get

$$T_S(-\log D)|_F \simeq \mathcal{O}_F \oplus \mathcal{O}_F(1).$$

Since $r = 6$ or 7, every component of $T_S(-\log D) \otimes \mathcal{L}^{\otimes(r-7)}|_F$ has degree > -2, so by Serre duality the desired H^2 vanishing holds.

The first row of the diagram therefore induces a surjection

$$H^1(S, T_S \otimes \mathcal{L}^{\otimes(r-7)}) \longrightarrow H^1(D, N_{D/S} \otimes \mathcal{L}^{\otimes(r-7)}).$$

By commutativity, it remains to show that the right vertical map also induces a surjection on H^1.

Write

$$\varphi^* f = s_D^r \alpha, \qquad \alpha \in H^0(S, \mathcal{L}^{\otimes(7-r)}), \qquad \alpha|_D \neq 0.$$

From the proof of Lemma 4.2, it follows that the right vertical map factors through

$$N_{D/S} \otimes \mathcal{L}^{\otimes(r-7)} \xrightarrow{r\alpha|_D} \mathcal{L}^{\otimes r}(-(r-1)D)|_D \hookrightarrow \mathcal{L}^{\otimes r}|_{rD}.$$

Identifying D with B and using $\mathcal{L}|_D \simeq K_B$, this factorization becomes

$$K_B^{\otimes(r-6)} \xrightarrow{r\alpha|_D} K_B \hookrightarrow \mathcal{L}^{\otimes r}|_{rD}.$$

The first map induces a surjection on H^1 since by Serre duality, its dual is the map

$$H^0(B, \mathcal{O}_B) \xrightarrow{r\alpha|_D} H^0(B, K_B^{\otimes(7-r)}),$$

which is injective because $\alpha|_D \neq 0$.

For the second map, note that $\mathcal{L}^{\otimes r}(-(r-1)D)|_D \simeq \mathcal{L}^{\otimes r} \otimes \frac{I_D^{r-1}}{I_D^r}$, the right hand side being the kernel of the natural map $\mathcal{L}^{\otimes r}|_{rD} \to \mathcal{L}^{\otimes r}|_{(r-1)D}$. Using the identification $\mathcal{L}|_D \simeq K_B$, we have the exact sequence

$$0 \longrightarrow K_B \longrightarrow \mathcal{L}^{\otimes r}|_{rD} \longrightarrow \mathcal{L}^{\otimes r}|_{(r-1)D} \longrightarrow 0.$$

The pushforward calculation above gives

$$H^1\big(S, \mathcal{L}^{\otimes r}|_{(r-1)D}\big) \simeq \bigoplus_{j=2}^{r} H^1(B, K_B^{\otimes j}) = 0,$$

since $g_B = 4$. Thus the second arrow also induces a surjection on H^1. This proves the required surjectivity and completes the proof. □

References

[1] F. Bastianelli, C. Ciliberto, F. Flamini, and P. Supino, *Cones of lines having high contact with general hypersurfaces and applications*, Mathematische Nachrichten, 296 (2023), pp. 509–522.

[2] R. Beheshti, *Linear subvarieties of hypersurfaces*, Int. Math. Res. Not., 49 (2005), pp. 3055–3063.

[3] ——, *Lines on projective hypersurfaces*, J. Reine Angew. Math., 592 (2006), pp. 1–21.

[4] R. Beheshti and E. Riedl, *Linear subspaces of hypersurfaces*, Duke Math. Journal, 10 (2021), pp. 2263–2288.

[5] A. Degtyarev, I. Itenberg, and J. C. Ottem, *Planes in cubic fourfolds*, Algebraic Geometry, 10 (2023), p. 228–258.

[6] G. A. Jennings, *Lines having high contact with a projective variety*, Pacific Journal of Mathematics, 125 (1986), pp. 103–115.

[7] J. Landsberg, *Lines on projective varieties*, J. Reine Angew. Math., 562 (2003), pp. 1–3.

[8] A. Patel, E. Riedl, G. Smith, and D. Tseng, *Lines highly tangent to a hypersurface*, Proceedings of the American Mathematical Society, 154 (2026), pp. 943—-956.

[9] A. Patel, E. Riedl, and D. Tseng, *Moduli of linear slices of high degree smooth hypersurfaces*, Algebra & Number Theory, 18 (2024), pp. 2133–2156.

[10] E. Rogora, *Varieties with many lines*, Manuscripta Math., 82 (1994), p. 207–226.

[11] E. Sernesi, *Deformations of Algebraic Schemes*, vol. 334 of Grundlehren der mathematischen Wissenschaften, Springer, Berlin, Heidelberg, 2006.

[12] J. M. Starr, *A fact about linear spaces on hypersurfaces*, Proceedings of the London Mathematical Society, 93 (2006), pp. 301–303. Appendix to T. D. Browning and D. R. Heath-Brown, "The density of rational points on non-singular hypersurfaces, II".

Department of Mathematics, Washington University in St. Louis, MO 63105
Email address: beheshti@wustl.edu

Department of Mathematics, Washington University in St. Louis, MO 63105
Email address: m.shibashis@wustl.edu