1. If $f(x, y) = x^2 \sin(y^2)$, then what is f_{xyx}?

(a) 0

(b) $2\sin(y^2)$

(c) $2\cos(y^2)$

(d) $4y\cos(y^2)$

(e) $y\cos(y^2)$

(f) $-2y\cos(y^2)$

$f_x = 2x \sin(y^2)$

$f_{xy} = 4xy \cos(y^2)$

$f_{xyx} = 4y \cos(y^2)$

2. What is an equation of the tangent plane to the surface

$$x^2 + y^3 + z^4 = 4$$

at $(2, -1, 1)$?

(a) $2x - y + z = 6$

(b) $2x + 3y + 4z = 5$

(c) $x + y + z = 1$

(d) $2x - y + z = 4$

(e) $4x - 2y + 3z = 11$

(f) $4x + 3y + 4z = 9$

$F(x,y,z) = x^2 + y^3 + z^4$

$\nabla F = \langle 2x, 3y^2, 4z^3 \rangle$

$\nabla F(2,-1,1) = \langle 4, 3, 4 \rangle$

so the equation of the tangent plane is:

$4(x-2) + 3(y+1) + 4(z-1) = 0$

$4x + 3y + 4z = 9$

3. Suppose that f is a differentiable function of x and y, and $x = g(t, s)$ and $y = h(t, s)$ are differentiable functions of t and s. If $g(1, 1) = 3$, $h(1, 1) = 2$, $f_x(3, 2) = 3$, $f_y(3, 2) = 4$, $g_t(1, 1) = -1$, and $h_t(1, 1) = 2$, then what is $\frac{\partial f}{\partial t}$ when $t = 1$ and $s = 1$?

(a) 2

(b) 11

(c) 5

(d) 16

(e) -4

(f) Impossible to know with the given information

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

$$\Rightarrow \frac{\partial f}{\partial t}(1,1) = \frac{\partial f}{\partial x}(3,2)\,\frac{\partial x}{\partial t}(1,1) + \frac{\partial f}{\partial y}(3,2)\,\frac{\partial y}{\partial t}(1,1)$$

$$= 3\,(-1) + 4\,(2) = 5$$

4. Suppose $f(x, y) = x^3y + x^2 + y^2$ and C is the curve of intersection of the plane $y = 2$ with the surface $z = f(x, y)$. Find the slope of the tangent line to C when $x = -1$.

(a) 3

(b) 0

(c) 4

(d) 1

(e) -1

(f) 2

slope is partial derivative with respect to x.

$f_x = 3x^2y + 2x$

$f_x(-1,2) = 6 - 2 = 4$

5. Suppose $f(x,y) = \sqrt{2+x^2+y^2}$. Use the linearization of $f(x,y)$ at $(1,1)$ to estimate $f(0.9, 1.2)$.

(a) 2

(b) 1.8

(c) 1.9

(d) 2.05

(e) 1.6

(f) 2.1

$$f_x = \frac{x}{\sqrt{2+x^2+y^2}} \qquad f_y = \frac{y}{\sqrt{2+x^2+y^2}}$$

$$f_x(1,1) = f_y(1,1) = \frac{1}{2}$$

$$L(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$$

$$= 2 + \frac{1}{2}(x-1) + \frac{1}{2}(y-1)$$

so $L(0.9, 1.2) = 2 + \frac{1}{2}(-0.1) + \frac{1}{2}(0.2) = 2.05$

6. Suppose that T is given by $x(e^y + e^{-y})$ where x and y are found to be 2 and $\ln 2$ with a possible of error of 0.2 in x and 0.1 in y. Using differentials, estimate the maximum possible error in the computed value of T.

(a) 0.1

(b) 0.2

(c) 0.4

(d) 0.8

(e) 1

(f) 2

$$dT = \frac{\partial T}{\partial x}\,dx + \frac{\partial T}{\partial y}\,dy$$

$$= \frac{5}{2} \times 0.2 + 3 \times 0.1$$

$$= 0.5 + 0.3$$

$$= 0.8$$

$$\frac{\partial T}{\partial x} = e^y + e^{-y}$$

$$\frac{\partial T}{\partial y} = x(e^y - e^{-y})$$

$$\Rightarrow \frac{\partial T}{\partial x}(2, \ln 2) = e^{\ln 2} + e^{-\ln 2}$$

$$= 2 + \frac{1}{2} = \frac{5}{2}$$

$$\frac{\partial T}{\partial y}(2, \ln 2) = 2(e^{\ln 2} - e^{-\ln 2})$$

$$= 2(2 - \frac{1}{2})$$

$$= 3$$

7. Suppose $\nabla f(3,6) = \langle -4, 6\rangle$ and for a unit vector $\mathbf{u}$, the directional derivative of f at $(3,6)$ in the direction of $\mathbf{u}$ is 0. Which of the following vectors is parallel to $\mathbf{u}$?

(a) $\langle -4, 6\rangle$

(b) $\langle 1, 2\rangle$

(c) $\langle -2, 1\rangle$

(d) $\langle 1, 1\rangle$

(e) $\langle 3, 2\rangle$

(f) No such unit vector exists.

$\nabla f \cdot \vec{u} = 0$, so u should be perpendicular to $\langle -4, 6\rangle$

8. Suppose f is a differentiable function of x, y and z. If $\nabla f(x_0, y_0, z_0) = \langle 2, 3, 6 \rangle$, then what is the minimum possible rate of change of f in any direction?

(a) -11

(b) -7

(c) 0

(d) $-\sqrt{21}$

(e) 1

(f) $-\dfrac{2}{3}$

minimum rate of change $= -|\nabla f| = -\sqrt{2^2+3^2+6^2}$

$= -\sqrt{49} = -7$

9. The level curve of $f(x, y) = \dfrac{2x^2 + 2 + 2y}{x^2 + y^2 + 1}$ which passes through $(1, 0)$ is

(a) a point

(b) a circle of radius 2

(c) two parallel lines

(d) two intersecting lines

(e) an ellipse

(f) a circle of radius 1

$f(1,0) = \dfrac{4}{2} = 2$

If $f(x,y) = 2$, then $\dfrac{2x^2 + 2 + 2y}{x^2 + y^2 + 1} = 2$, so

$2x^2 + 2 + 2y = 2x^2 + 2y^2 + 2$

so $y^2 = y$, so $y(y-1) = 0$ so $y = 1$ or $y = 0$

two parallel lines

10. Let $f(x, y) = x^2 + 4xy + xy^2$. Which of the following points is a critical point of f?

(a) $(0, 2)$

(b) $(-2, 0)$

(c) $(4, 4)$

(d) $(0, 4)$

(e) $(2, -2)$

(f) No critical points exist.

$f_x = 2x + 4y + y^2$

$f_y = 4x + 2xy$

11. Suppose

$$\mathbf{r}_1(t) = \langle t^2,\ t^3 - t^2 + t,\ 1 + t - t^2 \rangle$$

and

$$\mathbf{r}_2(s) = \langle s^2 - s + 1,\ e^s,\ s^3 + 2s + 1 \rangle.$$

If $\mathbf{r}_1$ and $\mathbf{r}_2$ are both contained on a surface S, then which of the following vectors is a normal vector for the tangent plane to S at $(1, 1, 1)$?

(a) $\langle 5, -3, 4 \rangle$

(b) $\langle 1, 1, 1 \rangle$

(c) $\langle 2, 2, -1 \rangle$

(d) $\langle -1, 1, 2 \rangle$

(e) $\langle 3, 6, 1 \rangle$

(f) $\langle 0, 1, 5 \rangle$

$\vec{r}_1'(t) = \langle 2t, 3t^2 - 2t + 1, 1 - 2t \rangle$ at $t = 1$ $\vec{r}_1(1) = \langle 1, 1, 1 \rangle$

So $\vec{r}_1'(1) = \langle 2, 2, -1 \rangle$

At $s = 0$ $\vec{r}_2(0) = \langle 1, 1, 1 \rangle$. $\vec{r}_2'(s) = \langle 2s - 1, e^s, 3s^2 + 2 \rangle$

So $\vec{r}_2'(0) = \langle -1, 1, 2 \rangle$

$\langle 2, 2, -1 \rangle \times \langle -1, 1, 2 \rangle = \langle 5, -3, 4 \rangle$

12. The function $f(x, y) = x^3 + xy^2 + y^2$ has exactly one critical point at $(0, 0)$. Find the maximum value of f on the closed disk $x^2 + y^2 \leq 1$.

(a) 0

(b) $\frac{5}{4}$

(c) 2

(d) 1

(e) $2\sqrt{3}$

(f) $3 - \sqrt{2}$.

$f(0,0) = 0$

on the circle $x^2+y^2=1$, we have

$$f(x,y) = x(x^2+y^2)+y^2 = x+y^2 = \underbrace{x+1-x^2}_{g(x)} \qquad -1 \leq x \leq 1$$

$g'(x) = 1-2x \Rightarrow$ critical point of g: $x = \frac{1}{2}$. $y = \pm\sqrt{1-\frac{1}{4}} = \pm\frac{\sqrt{3}}{2}$

$f(\frac{1}{2}, \frac{\sqrt{3}}{2}) = \frac{10}{8}$ ← maximum

$f(\frac{1}{2}, -\frac{\sqrt{3}}{2}) = \frac{10}{8}$

end points $\left\{ \begin{array}{l} f(1,0) = 1 \\ f(-1,0) = -1 \end{array} \right.$ ← minimum

13. Find the following limits or explain why the limit does not exist. (12 points)

(a) $\displaystyle\lim_{(x,y)\to(2,2)} \frac{x^2+xy-4x}{\sqrt{x+y}-2}$

$$\frac{x^2+xy-4x}{\sqrt{x+y}-2} = \frac{x(x+y-4)}{\sqrt{x+y}-2}\,\frac{\sqrt{x+y}+2}{\sqrt{x+y}+2}$$

$$= \frac{x(x+y-4)(\sqrt{x+y}+2)}{x+y-4} = x(\sqrt{x+y}+2)$$

So $\displaystyle\lim_{(x,y)\to(2,2)} \frac{x^2+xy-4x}{\sqrt{x+y}-2} = \lim_{(x,y)\to(2,2)} x(\sqrt{x+y}+2)$

$= 8$

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3y}{2x^4+y^4}$

The limit does not exist. We use the two path test

If $y=0$, then $\dfrac{x^3y}{2x^4+y^4} = \dfrac{0}{2x^4} = 0 \Rightarrow$ The limit is 0 when the function approaches $(0,0)$ along the x-axis.

If $x=y$, then $\dfrac{x^3y}{2x^4+y^4} = \dfrac{x^4}{3x^4} = \dfrac{1}{3} \Rightarrow$ the limit is $\frac{1}{3}$ when the function approaches $(0,0)$ along the line $y=x$.

14. (a) The function $f(x, y) = x^2 - xy^2 + y^2$ has critial points at $(0, 0)$, $(1, \sqrt{2})$, $(1, -\sqrt{2})$. Classify these critical points. (8 points)

$f_x = 2x - y^2 \qquad f_y = -2xy + 2y$

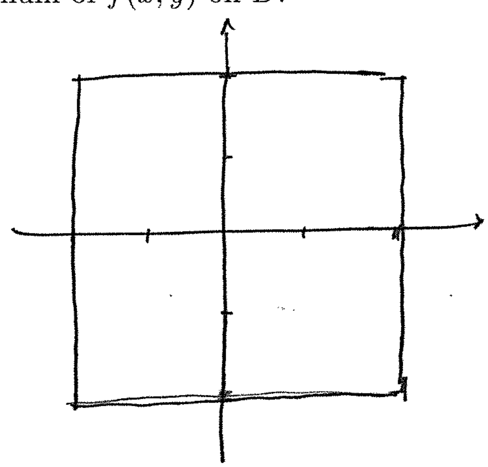
$f_{xx} = 2 \qquad f_{xy} = -2y \qquad f_{yy} = 2 \qquad D = \begin{vmatrix} 2 & -2y \\ -2y & 2 \end{vmatrix} = 4 - 4y^2$

At $(0,0)$ $\quad D(0,0) = 4 > 0 \quad f_{xx} > 0 \Rightarrow (0,0)$ is a local minimum.

At $(1,\sqrt{2})$ $\quad D(1,\sqrt{2}) = -4 < 0 \Rightarrow (1,\sqrt{2})$ is a saddle point.

At $(1,-\sqrt{2})$ $\quad D(1,-\sqrt{2}) = -4 < 0 \Rightarrow (1,-\sqrt{2})$ is a saddle point.

(b) Let D be the closed square region in the plane with vertices $(2, 2)$, $(-2, 2)$, $(2, -2)$, and $(-2, -2)$. Find the absolute maximum and minimum of $f(x, y)$ on D. (8 points)



critical points : $f(0,0) = \boxed{0}$

$f(1, \pm\sqrt{2}) = \boxed{1}$

boundary points : vertices :

$f(2, \pm 2) = 4 - 8 + 4 = \boxed{0}$ $\leftarrow$ min

$f(-2, \pm 2) = 4 + 8 + 4 = \boxed{16}$ $\leftarrow$ max

when $y = \pm 2$, $\quad f(x,y) = \underbrace{x^2 - 4x + 4}_{g(x)} \qquad g'(x) = 2x - 4 \Rightarrow g'(x) = 0$

when $x = 2$. We have already found $f(2, \pm 2)$.

- when $x=2$ $\quad f(x,y) = 4-2y^2+y^2 = \underbrace{4-y^2}_{h(y)}$ $\quad h'(y) = -2y$

$h'(y)=0$ when $y=0$.

$f(2,0) = \boxed{4}$

- when $x=-2$ $\quad f(x,y) = 4+2y^2+y^2 = \underbrace{4+y^2}_{h(y)}$ $\quad h'(y) = 2y$

So $h'(y)=0$ when $y=0$

$f(-2,0) = \boxed{4}$

minimum $= 0$

maximum $= 16$