Non-uniruledness results for spaces of rational curves in hypersurfaces

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Abstract

We prove that the sweeping components of the space of smooth rational curves in a smooth hypersurface of degree $d$ in $\mathbb{P}^n$ are not uniruled if \((n+1)/2 \leq d \leq n-3\). We also show that for any $e \geq 1$, the space of smooth rational curves of degree $e$ in a general hypersurface of degree $d$ in $\mathbb{P}^n$ is not uniruled roughly when $d \geq e\sqrt{n}$.

1 Introduction

Throughout this paper, we work over an algebraically closed field of characteristic zero $k$. Let $X$ be a smooth hypersurface of degree $d$ in $\mathbb{P}^n$, and for $e \geq 1$, let $R_e(X)$ denote the closure of the open subscheme of $\text{Hilb}_{e+1}(X)$ parametrizing smooth rational curves of degree $e$ in $X$. It is known that if $d < \frac{n+1}{2}$ and $X$ is general, then $R_e(X)$ is an irreducible variety of dimension $e(n+1-d)+n-4$, and it is conjectured that the same holds for general Fano hypersurfaces (see [6] and [2]). If $X$ is not general, $R_e(X)$ may be reducible. We call an irreducible component $R$ of $R_e(X)$ a sweeping component if the curves parametrized by its points sweep out $X$ or equivalently, if for a general curve $C$ parametrized by $R$, the normal bundle of $C$ in $X$ is globally generated.

If $d \leq n-1$, or if $d = n$ and $e \geq 2$, then $R_e(X)$ has at least one sweeping component.

In this paper, we study the birational geometry of sweeping components of $R_e(X)$. Recall that a projective variety $Y$ of dimension $m$ is called uniruled if there is a variety $Z$ of dimension $m-1$ and a dominant rational map $Z \times \mathbb{P}^1 \dashrightarrow Y$. We are interested in the following question: for which values of $n,d,$ and $e$, does $R_e(X)$ have non-uniruled sweeping components? Our original motivation for this study comes from the question of whether or not general Fano hypersurfaces of low indices are unirational.

We give a complete answer to the above question when $\frac{n+1}{2} \leq d \leq n-3$:

**Theorem 1.1.** Let $X$ be any smooth hypersurface of degree $d$ in $\mathbb{P}^n$, $(n+1)/2 \leq d \leq n-3$. Then for all $e \geq 1$, no sweeping component of $R_e(X)$ is uniruled.

We also consider the case $d = n-2$ and prove:

**Theorem 1.2.** Let $X$ be a smooth hypersurface of degree $n-2$ in $\mathbb{P}^n$, and let $C$ be a smooth rational curve of degree $e$ in $X$. Every irreducible sweeping component of $R_e(X)$ which contains $C$ is non-uniruled provided that when we split the normal bundle of $C$ in $\mathbb{P}^n$ as a sum of line bundles $N_C/\mathbb{P}^n = \mathcal{O}_C(a_1) \oplus \cdots \oplus \mathcal{O}_C(a_{n-1})$, we have $a_i + a_j < 3e$ for every $1 \leq i < j \leq n-1$. 

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When $n = 5$ and $d = 3$, $R_e(X)$ is irreducible for any smooth $X$ (see [2]). In [3], J. de Jong and J. Starr study the birational geometry of $R_e(X)$ with regards to the question of rationality of general cubic fourfolds. Let $\overline{\mathcal{M}}_{0,0}(X,e)$ be the Kontsevich moduli stack of stable maps of degree $e$ from curves of genus zero to $X$ and $\overline{\mathcal{M}}_{0,0}(X,e)$ the corresponding coarse moduli scheme. There is an open subscheme of $\overline{\mathcal{M}}_{0,0}(X,e)$ parametrizing smooth rational curves of degree $e$ in $X$. Presenting a general method to produce differential forms on desingularisations of $\overline{\mathcal{M}}_{0,0}(X,e)$, de Jong and Starr prove that if $X$ is a general cubic fourfold, then $R_e(X)$ is not uniruled when $e > 5$ is an odd integer, and the general fibers of the MRC fibration of a desingularization of $R_e(X)$ are at most 1-dimensional when $e > 4$ is an even integer.

If $X$ is a general cubic fourfold, then for a general rational curve $C$ of degree $e$ in $X$, the normal bundle of $C$ in $\mathbb{P}^5$ is isomorphic to $\mathcal{O}_C(\frac{3e-1}{2})^{\oplus 4}$ if $e \geq 5$ is odd and to $\mathcal{O}_C(\frac{3e-1}{2})^{\oplus 2} \oplus \mathcal{O}_C(\frac{3e-4}{2})^{\oplus 2}$ if $e \geq 6$ is an even integer (see [3, Proposition 7.1]). Thus Theorem 1.2 gives a new proof of the result of de Jong and Starr when $e \geq 5$ is odd. In section 4 we study the case when $e$ is an even integer and show:

**Theorem 1.3.** Let $X$ be a smooth cubic fourfold, and let $C$ be a general smooth rational curve of degree $e \geq 5$ in $X$.

- $R_e(X)$ is not uniruled if $e$ is odd and $N_{C/\mathbb{P}^5} = \mathcal{O}_C(\frac{3e-1}{2})^{\oplus 4}$.
- If $\tilde{R}$ is a desingularization of $R_e(X)$, then the general fibers of the MRC fibration of $\tilde{R}$ are at most 1-dimensional if $e$ is even and $N_{C/\mathbb{P}^5} = \mathcal{O}_C(\frac{3e-1}{2})^{\oplus 2} \oplus \mathcal{O}_C(\frac{3e-4}{2})^{\oplus 2}$.

It is an interesting question whether or not the splitting type of $N_{C/\mathbb{P}^5}$ is always as above for a general rational curve $C$ of degree $\geq 5$ in an arbitrary smooth cubic fourfold.

Finally, we consider the case $d < \frac{n+1}{2}$. When $d^2 \leq n$, $R_e(X)$ is uniruled. In fact, in this range a much stronger statement holds: for every $e \geq 2$, the space of based, 2-pointed rational curves of degree $e$ in $X$ is rationally connected in a suitable sense (see [4] and [11]). By [6], when $X$ is general and $d < \frac{n+1}{2}$, $\overline{\mathcal{M}}_{0,0}(X,e)$ is irreducible and therefore it is birational to $R_e(X)$. Starr [12] shows that if $d < \min(n-6, \frac{n+1}{2})$ and $d^2 + d \geq 2n + 2$, then for every $e \geq 1$, the canonical divisor of $\overline{\mathcal{M}}_{0,0}(X,e)$ is big. This suggests that when $d^2 + d \geq 2n + 2$ and $X$ is general, $R_e(X)$ may be non-uniruled. In Section 5, we show:

**Theorem 1.4.** Let $X \subset \mathbb{P}^n$ ($n \geq 12$) be a general hypersurface of degree $d$, and let $m \geq 1$ be an integer. If a general smooth rational curve $C$ in $X$ of degree $e$ is $m$-normal (that is if the global sections of $\mathcal{O}_{\mathbb{P}^n}(m)$ maps surjectively to those of $\mathcal{O}_{\mathbb{P}^n}(m)_{|C}$), and if

$$d^2 + (2m+1)d \geq (m+1)(m+2)n + 2,$$

then $R_e(X)$ is not uniruled.

In particular, since every smooth curve of degree $e \geq 3$ in $\mathbb{P}^n$ is $(e-2)$-normal, it follows that $R_e(X)$ is not uniruled when $X$ is general and

$$d^2 + (2e-3)d \geq e(e-1)n + 2.$$

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2 A Consequence of Uniruledness

In this section, we prove a proposition, analogous to the existence of free rational curves on non-singular uniruled varieties, for varieties whose spaces of smooth rational curves are uniruled.

We first fix notation and recall some definitions.

For a morphism \( f : Y \to X \) between smooth varieties, by the normal sheaf of \( f \) we will mean the cokernel of the induced map on the tangent bundles \( T_Y \to f^*T_X \).

If \( Y \) is an irreducible projective variety, and if \( \overline{Y} \) is a desingularization of \( Y \), then the maximal rationally connected (MRC) fibration of \( \overline{Y} \) is a smooth morphism \( \pi : Y^0 \to Z \) from an open subset \( Y^0 \subseteq \overline{Y} \) such that the fibers of \( \pi \) are all rationally connected, and such that for a very general point \( z \in Z \), any rational curve in \( \overline{Y} \) intersecting \( \pi^{-1}(z) \) is contained in \( \pi^{-1}(z) \). The MRC fibration of any smooth variety exists and is unique up to birational equivalences [9].

Let \( Y \) be an irreducible projective variety, and assume the fiber of the MRC fibration of \( \overline{Y} \) at a general point is \( m \)-dimensional. Then it follows from the definition that there is an irreducible component \( Z \) of \( \operatorname{Hom}(\mathbb{P}^1, Y) \) such that the map \( \mu_1 : Z \times \mathbb{P}^1 \to Y \) defined by \( \mu_1([g], b) = g(b) \) is dominant and the image of the map \( \mu_2 : Z \times \mathbb{P}^1 \times \mathbb{P}^1 \to Y \times Y \) defined by \( \mu_2([g], b_1, b_2) = (g(b_1), g(b_2)) \) has dimension \( \geq \dim Y + m \).

**Proposition 2.1.** Let \( X \subset \mathbb{P}^n \) be a nonsingular projective variety. If an irreducible sweeping component \( R \) of \( R_e(X) \) is uniruled, then there exist a smooth rational surface \( S \) with a dominant morphism \( \pi : S \to \mathbb{P}^1 \) and a generically finite morphism \( f : S \to X \) with the following two properties:

(i) If \( C \) is a general fiber of \( \pi \), then \( f|_C \) is a closed immersion onto a smooth curve parameterized by a general point of \( R \).

(ii) If \( N_f \) denotes the normal sheaf of \( f \), then \( \pi^*N_f \) is globally generated.

Moreover, if the fiber of the MRC fibration of a desingularization of \( R \) at a general point is at least \( m \)-dimensional, then there are such \( S \) and \( f \) with the additional property that \( \pi^*N_f \) has an ample subsheaf of rank \( m - 1 \).

**Proof.** Let \( U \subset R \times X \) be the universal family over \( R \). Since \( R \) is uniruled, there exist a quasi-projective variety \( Z \) and a dominant morphism \( \mu : Z \times \mathbb{P}^1 \to R \). Let \( V \subset Z \times \mathbb{P}^1 \times X \) be the pullback of the universal family to \( Z \times \mathbb{P}^1 \), and denote by \( q : V \to Z \times X \) and \( p : V \to Z \) the projection maps.

Consider a desingularization \( g : \overline{V} \to V \), and let \( \overline{q} = g \circ q \) and \( \overline{p} = p \circ g \). Let \( z \in Z \) be a general point, and denote the fibers of \( p \) and \( \overline{p} \) over \( z \) by \( S \) and \( \overline{S} \) respectively. Let \( f : S \to X \) be the restriction of \( q \) to \( S \), and let \( \overline{f} = f \circ g : \overline{S} \to X \). Since \( z \) is general, by generic smoothness, \( \overline{S} \) is a smooth surface whose general fiber over \( \mathbb{P}^1 \) is a smooth connected rational curve. We claim that \( \overline{S} \) and \( \overline{f} \) satisfy the desired properties. The first property is clearly satisfied.

Since every coherent sheaf on \( \mathbb{P}^1 \) splits as a torsion sheaf and a direct sum of line bundles, to show that \( \pi^*N_f \) is globally generated, it suffices to check that the restriction map \( H^0(\mathbb{P}^1, \pi^*N_f) \to N_f|_b \) is surjective for a general point \( b \in \mathbb{P}^1 \), or equivalently, that the restriction map \( H^0(S, N_f) \to H^0(C, N_f|_C) \) is surjective for a general fiber \( C \). To show this, we consider the Kodaira-Spencer map associated to \( \overline{V} \) at a general point \( z \in Z \). Denote by \( N_{\overline{q}} \) the normal sheaf of the map \( \overline{q} \). We get a sequence of maps

\[
T_{Z,z} \to H^0(\overline{S}, \overline{p}^*T_Z|_{\overline{S}}) \to H^0(\overline{S}, T_{\overline{S} \times X,z}|_{\overline{S}}) \to H^0(\overline{S}, N_{\overline{q}}|_{\overline{S}}).
\]

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Let \( b \) be a general point of \( \mathbf{P}^1 \). Composing the above map with the projection map \( T_{Z \times \mathbf{P}^1,(z,b)} \to T_{Z,z} \), we get a map \( T_{Z \times \mathbf{P}^1,(z,b)} \to H^0(S, N_{\tilde{f}/Z}) \). Note that if \( N_{\tilde{f}} \) denotes the normal sheaf of \( \tilde{f} \), then \( N_{\tilde{f}|Z} \) is naturally isomorphic to \( N_{\tilde{f}'} \). Also, if \( C \) is the fiber of \( \pi : \tilde{S} \to \mathbf{P}^1 \) over \( b \), then since \( b \) is general, \( C \) is smooth, and we have a short exact sequence

\[
0 \to N_{C/\tilde{S}} \to N_{\tilde{f}(C)/X} \to N_{\tilde{f}|C} \to 0.
\]

So we get a commutative diagram

\[
\begin{array}{ccc}
T_{Z \times \mathbf{P}^1,(z,b)} & \longrightarrow & T_{Z,z} \\
\downarrow d\mu_{(z,b)} & & \downarrow \\
T_{R,[\tilde{f}(C)]} = H^0(\tilde{f}(C), N_{\tilde{f}(C)/X}) & \longrightarrow & H^0(C, N_{\tilde{f}|C})
\end{array}
\]

Since \( \mu \) is dominant, and since \( R \) is sweeping and therefore generically smooth, \( d\mu_{(z,b)} \) is surjective. Since the bottom row is also surjective, the map \( H^0(S, N_{\tilde{f}}) \to H^0(C, N_{\tilde{f}|C}) \) is surjective as well. Thus \( \tilde{\pi}_* N_{\tilde{f}} \) is globally generated.

Suppose now that \( R \) is uniruled and that the general fibers of the MRC fibration of \( R \) are at least \( m \)-dimensional. Let \( \dim R = r \). Then there exists a morphism \( \mu_1 : Z \times \mathbf{P}^1 \to R \) such that the image of

\[
\mu_2 : Z \times \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{R} \times \mathbf{R}
\]

\[
\mu_2(z, b_1, b_2) = (\mu_1(z, b_1), \mu_1(z, b_2))
\]

has dimension \( \geq r + m \). Constructing \( \tilde{S} \) and \( \tilde{f} \) as before, and if \( C_1 \) and \( C_2 \) denote the fibers of \( \pi \) over general points \( b_1 \) and \( b_2 \) of \( \mathbf{P}^1 \), then the image of the map

\[
d\mu_2 : T_{Z \times \mathbf{P}^1 \times \mathbf{P}^1,(z,b_1,b_2)} \to T_{R \times R, ([\tilde{f}(C_1)], [\tilde{f}(C_2)])} = H^0(C_1, N_{\tilde{f}(C_1)/X}) \oplus H^0(C_2, N_{\tilde{f}(C_2)/X})
\]

is at least \( (r + m) \)-dimensional. The desired result now follows from the following commutative diagram

\[
\begin{array}{ccc}
T_{Z \times \mathbf{P}^1 \times \mathbf{P}^1,(z,b_1,b_2)} & \longrightarrow & T_{Z,z} \\
\downarrow (d\mu_2),(z,b_1,b_2) & & \downarrow \\
T_{R \times R, ([\tilde{f}(C_1)], [\tilde{f}(C_2)])} & \longrightarrow & H^0(C_1, N_{\tilde{f}(C_1)}) \oplus H^0(C_2, N_{\tilde{f}(C_2)})
\end{array}
\]

and the observation that the kernel of the bottom row is 2-dimensional. \( \square \)

The above proposition will be enough for the proof of Theorem 1.1, but to prove Theorem 1.3 in the even case, we will need a slightly stronger variant. Let \( f : Y \to X \) be a morphism between smooth varieties, and let \( N_f \) be the normal sheaf of \( f \)

\[
0 \to T_Y \to f^* T_X \to N_f \to 0.
\]
Suppose there is a dominant map $\pi : Y \to \mathbb{P}^1$, and let $M$ be the image of the map induced by $\pi$ on the tangent bundles $T_Y \to \pi^*T_{\mathbb{P}^1}$. Consider the push-out of the above sequence by the map $T_Y \to M$

\[
\begin{array}{ccccccc}
0 & \longrightarrow & T_Y & \longrightarrow & f^*T_X & \longrightarrow & N_f & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & M & \longrightarrow & N_{f,\pi} & \longrightarrow & N_f & \longrightarrow & 0 \\
\end{array}
\]

The sheaf $N_{f,\pi}$ in the above diagram will be referred to as the normal sheaf of $f$ relative to $\pi$.

Property (ii) of Proposition 2.1 says that $H^0(S, N_f) \to H^0(C, (N_f|_C))$ is surjective. An argument parallel to the proof of Proposition 2.1 shows the following:

**Proposition 2.2.** Let $X$ be as in Proposition 2.1. Then property (ii) can be strengthened as follows:

(ii') If $N_f$ denotes the normal sheaf of $f$, and if $N_{f,\pi}$ denotes the normal sheaf of $f$ relative to $\pi$, then the composition of the maps

$H^0(S, N_{f,\pi}) \to H^0(C, N_{f,\pi}|_C) \to H^0(C, N_f|_C)$

is surjective for a general fiber $C$ of $\pi$.

Moreover, if the general fibers of the MRC fibration of a desingularization of $R$ are at least $m$-dimensional, then there are $S$ and $f$ with properties (i) and (ii') such that the image of the map

$H^0(S, N_{f,\pi} \otimes I_C) \to H^0(C, (N_f \otimes I_C)|_C)$

is at least $(m - 1)$-dimensional.

### 3 The case when $\frac{n+1}{2} \leq d$

Let $X$ be a smooth hypersurface of degree $d$ in $\mathbb{P}^n$. Assume that a sweeping component $R$ of $R_e(X)$ is uniruled. The following result, along with Proposition 2.1 will prove Theorem 1.1.

**Proposition 3.1.** Suppose $d \leq n - 3$, and let $S$ and $f$ be as in Proposition 2.1. If $C$ is a general fiber of $\pi : S \to \mathbb{P}^1$ and $I_C$ is the ideal sheaf of $C$ in $S$, then the restriction map

$H^0(S, f^*\mathcal{O}_X(2d - n - 1) \otimes I_C^\vee) \to H^0(C, f^*\mathcal{O}_X(2d - n - 1) \otimes I_C^\vee|_C)$

is zero.

**Proof of Theorem 1.1.** Granting Proposition 3.1, since $H^0(S, f^*\mathcal{O}_X(2d-n-1)\otimes I_C^\vee) \to H^0(C, f^*\mathcal{O}_X(2d-n-1) \otimes I_C^\vee|_C)$ is the zero map, we have

$H^0(S, f^*\mathcal{O}_X(2d - n - 1)) = H^0(S, f^*\mathcal{O}_X(2d - n - 1) \otimes I_C^\vee).$
Thus,

\[ H^0(\mathbb{P}^1, \pi_* f^* O_X(2d - n - 1)) = H^0(\mathbb{P}^1, \pi_*(f^* O_X(2d - n - 1) \otimes I^\cap)) \]

\[ = H^0(\mathbb{P}^1, (\pi_* f^* O_X(2d - n - 1)) \otimes O_{\mathbb{P}^1}(1)), \]

which is only possible if \( H^0(\mathbb{P}^1, \pi_* f^* O_X(2d - n - 1)) = 0 \) and \( d < (n + 1)/2 \).

**Proof of Proposition 3.1.** Let \( \omega_S \) be the canonical sheaf of \( S \). By Serre duality and the long exact sequence of cohomology, it suffices to show that if \( S \) and \( f \) satisfy the properties of Proposition 2.1, then the restriction map

\[ H^1(S, f^* O_X(n + 1 - 2d) \otimes \omega_S) \to H^1(C, f^* O_X(n + 1 - 2d) \otimes \omega_S|_C) \]

is surjective. Let \( N \) be the normal sheaf of the map \( f : S \to X \), and let \( N' \) be the normal sheaf of the map \( S \to \mathbb{P}^n \). There is a short exact sequence

\[ 0 \to N \to N' \to f^* O_X(d) \to 0. \tag{1} \]

Taking the \((n - 3)\)-rd exterior power of this sequence, we get the following short exact sequence

\[ 0 \to \bigwedge^{n-3} N \otimes f^* O_X(-d) \to \bigwedge^{n-3} N' \otimes f^* O_X(-d) \to \bigwedge^{n-4} N \to 0. \]

For an exact sequence of sheaves of \( O_S \)-modules \( 0 \to E \to F \to M \to 0 \) with \( E \) and \( F \) locally free of ranks \( e \) and \( f \), there is a natural map of sheaves \( \bigwedge^{f-e-1} M \otimes \bigwedge^e E \otimes (\bigwedge^f F)^\vee \to M^\vee \) which is defined locally at a point \( s \in S \) as follows: assume \( \gamma_1, \ldots, \gamma_{f-e-1} \in M_s, \alpha_1, \ldots, \alpha_e \in E_s \), and \( \phi : \bigwedge^f F_s \to O_{S,s} \); then for \( \gamma \in M_s \), we set \( \gamma_{f-e} = \gamma \), and we define the map to be \( \gamma \mapsto \phi(\tilde{\gamma}_1 \wedge \tilde{\gamma}_2 \wedge \cdots \wedge \tilde{\gamma}_{f-e} \wedge \alpha_1 \wedge \cdots \wedge \alpha_e) \) where \( \tilde{\gamma}_i \) is any lifting of \( \gamma_i \) in \( F_s \). Clearly, this map does not depend on the choice of the liftings, and thus it is defined globally. So from the short exact sequence \( 0 \to T_S \to f^* T_X \to N \to 0 \), we get a map

\[ \bigwedge^{n-4} N \to N^\vee \otimes f^* O_X(n + 1 - d) \otimes \omega_S, \]

and from the short exact sequence \( 0 \to T_S \to f^* T_{\mathbb{P}^n} \to N' \to 0 \), we get a map

\[ \bigwedge^{n-3} N' \otimes f^* O_X(-d) \to (N')^\vee \otimes f^* O_X(n + 1) \otimes \omega_S. \]

With the choices of the maps we have made, the following diagram, whose bottom row is obtained from dualizing sequence (1) and tensoring with \( f^* O_X(n + 1 - 2d) \otimes \omega_S \), is commutative with exact rows

\[
\begin{array}{ccccccccc}
0 & \to & \bigwedge^{n-3} N \otimes f^* O_X(-d) & \to & \bigwedge^{n-3} N' \otimes f^* O_X(-d) & \to & \bigwedge^{n-4} N & \to & 0 \\
0 & \to & f^* O_X(n + 1 - 2d) \otimes \omega_S & \to & (N')^\vee \otimes f^* O_X(n + 1 - d) \otimes \omega_S & \to & N^\vee \otimes f^* O_X(n + 1 - d) \otimes \omega_S & \to & 0 \\
\end{array}
\]
Since the cokernel of the first vertical map restricted to \( C \) is a torsion sheaf, to show the assertion, it suffices to show that the map
\[
H^1(S, \bigwedge^{n-3} N \otimes f^* \mathcal{O}_X(-d)) \to H^1(C, \bigwedge^{n-3} N \otimes f^* \mathcal{O}_X(-d)|_C)
\]
is surjective. Applying the long exact sequence of cohomology to the top sequence, the surjectivity assertion follows if we show that

1) \( H^0(S, \bigwedge^{n-4} N) \to H^0(C, \bigwedge^{n-4} N|_C) \) is surjective,

2) \( H^1(C, \bigwedge^{n-3} N' \otimes f^* \mathcal{O}_X(-d)|_C) = 0. \)

To prove (1), we consider the commutative diagram
\[
\begin{array}{ccc}
\bigwedge^{n-4} H^0(S, N) & \longrightarrow & \bigwedge^{n-4} H^0(C, N|_C) \\
\downarrow & & \downarrow \\
H^0(S, \bigwedge^{n-4} N) & \longrightarrow & H^0(C, \bigwedge^{n-4} N|_C).
\end{array}
\]
The top horizontal map is surjective since \( H^0(S, N) \to H^0(C, N|_C) \) is surjective, and the right vertical map is surjective since \( N|_C \) is a globally generated line bundles over \( \mathbb{P}^1 \). By commutativity of the diagram the bottom horizontal map is surjective.

To prove (2), note that there is a surjective map \( f^* \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n+1} \to N' \). Taking the \((n-3)\)-rd exterior power, and then tensoring with \( f^* \mathcal{O}_X(-d) \), we get a surjective map
\[
f^* \mathcal{O}_{\mathbb{P}^n}(n-3-d)^{\oplus(n+1)} \to \bigwedge^{n-3} N' \otimes f^* \mathcal{O}_X(-d).
\]
Restricting to \( C \), since \( n-3-d \geq 0 \), we have \( H^1(C, \bigwedge^{n-3} N' \otimes f^* \mathcal{O}_X(-d)|_C) = 0. \)

\[\square\]

**Proof of Theorem 1.2.** Suppose that \( X \) is a smooth hypersurface of degree \( n-2 \) in \( \mathbb{P}^n \). Let \( C \) be a smooth rational curve of degree \( e \) in \( \mathbb{P}^n \) whose normal bundle \( N_{C/\mathbb{P}^n} \) is globally generated. If we write
\[
N_{C/\mathbb{P}^n} = \mathcal{O}_C(a_1) \oplus \cdots \oplus \mathcal{O}_C(a_{n-1}),
\]
then \( \sum_{1 \leq i \leq n-1} a_i = e(n+1) - 2 \). Assume that \( a_i + a_j < 3e \) for every \( 1 \leq i < j \leq n-1 \). Then \( H^1(C, \bigwedge^{n-3} N_{C/\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}(-d)|_C) = 0 \), and so if \( N' \) is as in the proof of Theorem 1.1, then
\[
H^1(C, \bigwedge^{n-3} N' \otimes f^* \mathcal{O}_X(-d)|_C) = 0.
\]
The assertion now follows from the proof of Theorem 1.1. \[\square\]

We remark that when \( d = n - 1 \) or \( n \), the uniruledness of the sweeping subvarieties of \( R_e(X) \) has been studied in [1]. It is proved that if \( e \leq n \), then a subvariety of \( R_e(X) \) is non-uniruled if the curves parametrized by its points sweep out \( X \) or a divisor in \( X \).
4 Cubic Fourfolds

In this section we prove Theorem 1.3. When \( e \geq 5 \) is odd, the theorem follows from Theorem 1.2 and [3, Proposition 7.1]. So let \( e \geq 6 \) be an even integer, and assume to the contrary that the general fibers of the MRC fibration of \( R_e(X) \) are at least 2-dimensional. Let \( S \) and \( f \) be as in Proposition 2.2, and let \( C \) be a general fiber of \( \pi \). Set \( N = N_f \) and \( Q = N_f, \pi \). Then by Proposition 2.1 the following properties are satisfied:

- Property (i): The composition of the maps
  \[
  H^0(S, Q) \to H^0(S, Q|_C) \to H^0(C, N|_C)
  \]
  is surjective.

- Property (ii): The composition of the maps
  \[
  H^0(S, Q \otimes I_C) \to H^0(C, Q \otimes I_C|_C) \to H^0(C, N \otimes I_C|_C)
  \]
  is non-zero.

We show these lead to a contradiction. Note that \( I_C|_C \) is isomorphic to the trivial bundle \( O_C \), but we write \( I_C|_C \) instead of \( O_C \) to keep track of various maps and exact sequences involved in the proof.

Let \( Q' \) be the normal sheaf of the map \( S \to \mathbf{P}^5 \) relative to \( \pi \). We have \( Q|_C = N_{C/X} \) and \( Q'|_C = N_{C/P^5} \). Since \( N_{X/P^5} = O_X(3) \), there is a short exact sequence

\[
0 \to Q \to Q' \to f^*O_X(3) \to 0. \tag{2}
\]

Taking exterior powers, we obtain the following short exact sequence

\[
0 \to \bigwedge^2 Q \otimes f^*O_X(-3) \to \bigwedge^2 Q' \otimes f^*O_X(-3) \to Q \to 0. \tag{3}
\]

Since this sequence splits locally, its restriction to \( C \) is also a short exact sequence

\[
0 \to \bigwedge^2 Q \otimes f^*O_X(-3)|_C \to \bigwedge^2 Q' \otimes f^*O_X(-3)|_C \to Q|_C \to 0. \tag{4}
\]

To get a contradiction, we show that the image of the boundary map

\[
\gamma : H^0(C, Q|_C) \to H^1(C, \bigwedge^2 Q \otimes f^*O_X(-3)|_C)
\]

is of codimension at least 2 in \( H^1(C, \bigwedge^2 Q \otimes f^*O_X(-3)|_C) \). This is not possible since by our assumption

\[
N_{C/P^5} = O_C(\frac{3e}{2}) \oplus O_C(\frac{3e}{2} - 1) \oplus O_C(\frac{3e}{2} - 2),
\]

and so

\[
H^1(C, \bigwedge^2 Q' \otimes f^*O_X(-3)|_C) = H^1(C, \bigwedge^2 N_{C/P^5} \otimes f^*O_X(-3)|_C) = H^1(C, O_C(-2) \oplus O_C(-1) \oplus O_C)
\]

\[= \mathbf{k}.\]
Lemma 4.1. The kernel of the map \( f^*T_X \to Q \) is a line bundle which contains \( \bigwedge^2 T_S \otimes \pi^*\omega_{\mathbb{P}^1} \) as a subsheaf.

Proof. The kernel of \( f^*T_X \to Q \) is equal to the kernel of the map induced by \( \pi \) on the tangent bundles \( T_S \to \pi^*T_{\mathbb{P}^1} \) which we denote by \( F \)

\[
0 \to F \to T_S \to \pi^*T_{\mathbb{P}^1}.
\]

Since \( F \) is reflexive, it is locally free on \( S \), and it is clearly of rank 1. Also the composition of the maps

\[
\bigwedge^2 T_S \otimes \pi^*\omega_{\mathbb{P}^1} \to \bigwedge^2 T_S \otimes \Omega_S = T_S \to \pi^*T_{\mathbb{P}^1}
\]

is the zero-map. So \( \bigwedge^2 T_S \otimes \pi^*\omega_{\mathbb{P}^1} \) is a subsheaf of \( F \).

Given a section \( r \in H^0(C, Q \otimes I_C|_C) \), we can define a map

\[
\beta_r : H^1(C, \bigwedge^2 Q \otimes f^*\mathcal{O}_X(-3)|_C) \to H^1(C, \omega_S|_C) = \mathbb{k}
\]
as follows. Let \( F \) be the line bundle from the proof of Lemma 4.1. It follows from the proof of the lemma that there is an injection \( \bigwedge^2 T_S \otimes \pi^*\omega_{\mathbb{P}^1} \to F \), and from the short exact sequence

\[
0 \to F \to f^*T_X \to Q \to 0
\]
we get a generically injective map of sheaves

\[
\bigwedge^3 Q \otimes F \to \bigwedge^4 f^*T_X.
\]

Combining these, we get a morphism

\[
\bigwedge^3 Q \otimes (\omega_S \otimes \pi^*T_{\mathbb{P}^1})^\vee \to \bigwedge^4 f^*T_X.
\]

Since \( \bigwedge^4 f^*T_X = f^*\mathcal{O}_X(3) \), we get a generically injective map

\[
\Psi : \bigwedge^3 Q \otimes f^*\mathcal{O}_X(-3) \otimes I_C \to \omega_S \otimes \pi^*T_{\mathbb{P}^1} \otimes I_C,
\]

and by restricting to \( C \), we get a map

\[
\Psi|_C : (\bigwedge^3 Q \otimes f^*\mathcal{O}_X(-3) \otimes I_C)|_C \to \omega_S|_C.
\]

Finally, \( r \) gives a map

\[
\Phi_r : \bigwedge^2 Q \otimes f^*\mathcal{O}_X(-3)|_C \xrightarrow{\Lambda r} \bigwedge^3 Q \otimes f^*\mathcal{O}_X(-3) \otimes I_C|_C,
\]

and we define \( \beta_r \) to be the map induced by the composition \( \Psi|_C \circ \Phi_r \). Note that \( \beta_r \) is non-zero if \( r \neq 0 \).
Lemma 4.2. For \( r, r' \in H^0(C, Q \otimes I_C|_C) \), \( \ker(\beta_r) = \ker(\beta_{r'}) \) if and only if \( r \) and \( r' \) are scalar multiples of each other.

Proof. By Serre duality, it is enough to show that the images of the maps

\[
H^0(C, I_C^r|_C) = H^0(C, \omega_C^r|_C \otimes \omega_C) \xrightarrow{\beta_r^\vee} H^0(C, (\bigwedge^2 Q^r \otimes f^*\mathcal{O}_X(3))|_C \otimes \omega_C)
\]

are the same if and only if \( r \) and \( r' \) are scalar multiples of each other. Since \( Q|_C = N_{C/X} \), we have \( \bigwedge^3 Q|_C = \bigwedge^3 N_{C/X} = f^*\mathcal{O}_X(3) \otimes \omega_C \), so

\[
(\bigwedge^2 Q^r \otimes f^*\mathcal{O}_X(3))|_C \otimes \omega_C = Q|_C,
\]

and the map

\[
\beta_r^\vee : H^0(C, I_C^r|_C) \to H^0(C, Q|_C)
\]

is simply given by \( r \). Similarly, \( \beta_r^\vee \) is given by \( r' \), and the lemma follows. \( \square \)

Recall that by definition, we have a short exact sequence

\[
0 \to \pi^*\mathcal{T}_{P^1}|_C \to Q|_C \to N|_C \to 0,
\]

and \( \pi^*\mathcal{T}_{P^1}|_C = I_C^{-1}|_C \). If we tensor this sequence with \( I_C|_C \), we get the following short exact sequence

\[
0 \to \mathcal{O}_C \to Q \otimes I_C|_C \to N \otimes I_C|_C \to 0.
\]

Let \( i \) be a non-zero section in the image of \( H^0(C, \mathcal{O}_C) \to H^0(C, Q \otimes I_C|_C) \). Then \( i \) induces a map

\[
\beta_i : H^1(C, \bigwedge^2 Q \otimes f^*\mathcal{O}_X(-3)|_C) \to H^1(C, \omega_S|_C) = k
\]

as described before. Let

\[
\gamma : H^0(C, Q|_C) \to H^1(C, \bigwedge^2 Q \otimes f^*\mathcal{O}_X(-3)|_C)
\]

be the connecting map in sequence (4).

Lemma 4.3. We have image(\( \gamma \)) \( \subset \ker \beta_i \).

Proof. Since the short exact sequence \( 0 \to N \to N' \to f^*\mathcal{O}_X(3) \to 0 \) splits locally, there is an exact sequence

\[
0 \to \bigwedge^2 N \otimes f^*\mathcal{O}_X(-3) \to \bigwedge^2 N' \otimes f^*\mathcal{O}_X(-3) \to N \to 0.
\]

Applying the long exact sequence of cohomology to the restriction of this sequence to \( C \), we get a map

\[
H^0(C, N|_C) \to H^1(C, \bigwedge^2 N \otimes f^*\mathcal{O}_X(-3)|_C).
\]
Also from the the exact sequence \(0 \to T_S \to f^*T_X \to N \to 0\), we get a map \(\wedge^2 T_S \otimes \wedge^2 N \to \wedge^4 f^*T_X = f^*\mathcal{O}_X(3)\) and hence a map
\[
\wedge^2 N \otimes f^*\mathcal{O}_X(-3) \to \omega_S.
\]
It follows from the definition of \(\beta_i\) that the map \(\beta_i \circ \gamma\) factors through
\[
H^0(C, Q|_C) \to H^0(C, N|_C) \to H^1(C, \wedge^2 N \otimes f^*\mathcal{O}_X(-3)|_C) \to H^1(C, \omega_S|_C),
\]
so we have a commutative diagram
\[
\begin{array}{c}
H^0(S, N) \to H^1(S, \wedge^2 N \otimes f^*\mathcal{O}_X(-3)) \to H^1(S, \omega_S) = 0 \\
| \\
H^0(C, Q|_C) \to H^0(C, N|_C) \to H^1(C, \omega_S|_C).
\end{array}
\]
Thus we can conclude the assertion from the fact that the restriction map \(H^0(S, N) \to H^0(C, N|_C)\) is surjective, and so the image of the composition of the above maps is contained in the image of the restriction map \(H^1(S, \omega_S) \to H^1(C, \omega_S|_C)\) which is zero.

In the following lemma we prove a similar result for the sections of \(Q \otimes I_C|_C\) which are obtained by restricting the global sections of \(Q \otimes I_C\) to \(C\).

**Lemma 4.4.** If \(\tilde{r} \in H^0(S, Q \otimes I_C)\), and if \(r = \tilde{r}|_C\), then \(\text{image}(\gamma) \subset \ker(\beta_r)\).

**Proof.** We have a commutative diagram
\[
\begin{array}{c}
H^0(S, Q) \to H^1(S, \wedge^2 Q \otimes f^*\mathcal{O}_X(-3)) \to H^1(S, \omega_S) = 0 \\
| \\
H^0(C, Q|_C) \xrightarrow{\gamma} H^1(C, \wedge^2 Q \otimes f^*\mathcal{O}_X(-3)) \xrightarrow{\beta_r} H^1(C, \omega_S|_C)
\end{array}
\]
and therefore for any \(u \in H^0(C, Q|_C)\) in the image of the restriction map \(H^0(S, Q) \to H^0(C, Q|_C)\), we have \(\beta_r(\gamma(u)) = 0\). Consider the exact sequence
\[
0 \to I_C^{-1}|_C \to Q|_C \to N|_C \to 0.
\]
From the hypothesis that the composition map \(H^0(S, Q) \to H^0(C, Q|_C) \to H^0(C, N|_C)\) is surjective, we see that to prove the statement, it is enough to show that for any non-zero \(u\) in the image of \(H^0(C, I_C^{-1}|_C) \to H^0(C, Q|_C)\), we have \(\gamma(u) \in \ker(\beta_r)\).

Consider the diagram
\[
\begin{array}{c}
H^0(C, Q|_C) \xrightarrow{\gamma} H^1(C, \wedge^2 Q \otimes f^*\mathcal{O}_X(-3)|_C) \\
\wedge \bigg\downarrow \wedge \bigg\downarrow \wedge \bigg\downarrow \wedge \bigg\downarrow \\
H^0(C, \wedge^2 Q \otimes I_C|_C) \xrightarrow{\chi} H^1(C, \wedge^3 Q \otimes f^*\mathcal{O}_X(-3) \otimes I_C|_C) \xrightarrow{\psi} H^1(C, \omega_S|_C)
\end{array}
\]
where $\lambda$ is obtained from applying the long exact sequence of cohomology to the third wedge power of sequence (2), and $\psi$ is induced by the map $\Psi|_C$. Then we have
\[
\beta_r \circ \gamma(u) = \psi \circ \lambda(u \wedge r) = \psi \circ \lambda(r \wedge i) \quad \text{(up to a scalar factor)}
\]
\[
= \beta_r \circ \gamma(r)
\]
\[
= 0,
\]
where the last equality comes from the fact that $\gamma(H^0(C, Q_C)) \subset \ker \beta_i$ by Lemma 4.3.

Let now $\tilde{r}_0 \in H^0(S, Q \otimes I_C)$ be so that its image in $H^0(C, N \otimes I_C|_C)$ is non-zero. Such $\tilde{r}_0$ exists by Property (ii). Then $r_0 := \tilde{r}_0|_C$ defines a map $\beta_{r_0}$. Since the image of $r_0$ in $H^0(C, N \otimes I_C|_C)$ is non-zero, $r_0$ and $i$ are not scalar multiples, so according to Lemma 4.2, $\ker \beta_{r_0} \neq \ker \beta$. Thus the codimension of $\ker \beta_i \cap \ker \beta_{r_0}$ is at least 2. On the other hand, by the previous lemmas, $\text{image}(\gamma) \subset \ker \beta_i \cap \ker \beta_{r_0}$. This is a contradiction since $\dim H^1(C, \wedge^2 Q^* \otimes f^* O_X(-3)|_C) = 1$.

5 The case when $d < \frac{n+1}{2}$

Throughout this section, $X \subset \mathbb{P}^n$ will be a general hypersurface of degree $d < (n+1)/2$. By the main theorem of [6], $R_e(X)$ is irreducible for every $e \geq 1$. If $d^2 \leq n$ and $e \geq 2$, then by [4] and [11], the space of rational curves of degree $e$ in $X$ passing through two general points of $X$ is rationally connected. In particular, $R_e(X)$ is rationally connected for $e \geq 2$. If $e = 1$, then $R_1(X)$ is the Fano variety of lines in $X$ which is rationally connected if and only if $d^2 + d \leq 2n$ [8, V.4.7]. In this section, we will consider the case when $d^2 + d > 2n$.

Assume that $R_e(X)$ is uniruled. Then there are $S$ and $f$ with the two properties given in Proposition 2.1. We can take the pair $(S, f)$ to be minimal in the sense that a component of a fiber of $\pi$ which is contracted by $f$ cannot be blown down. Let $N$ be the normal sheaf of $f$, and let $C$ be a general fiber of $\pi$ with ideal sheaf $I_C$ in $S$. Denote by $H$ the pullback of a hyperplane in $\mathbb{P}^n$ to $S$, and denote by $K$ a canonical divisor on $S$. From the exact sequences $0 \to T_S \to f^* T_X \to N \to 0$ and $0 \to f^* T_X \to f^* T_{\mathbb{P}^n} \to f^* O_{\mathbb{P}^n}(d) \to 0$ we get
\[
\chi(N \otimes I_C) = (n + 1) \chi(f^* O_{\mathbb{P}^n}(1) \otimes I_C) - \chi(f^* O_{\mathbb{P}^n}(d) \otimes I_C) - \chi(T_S \otimes I_C)
\]
\[
= (n + 1) \left( \frac{(H - C) \cdot (H - C - K)}{2} + 1 \right) - (dH - C) \cdot (dH - C - K) - 1
\]
\[
- \frac{-C \cdot (-C - K)}{2} - 1 - (2K^2 - 14)
\]
\[
= \frac{(n + 1 - d^2)H^2 - (n + 1 - d)H \cdot K - 2K^2 - (n + 1 - d)e + 14}{2}.
\]

We claim that $2H + 2C + K$ is base-point free and hence has a non-negative self-intersection number. By the main theorem of [10], if $2H + 2C + K$ is not base point free, then there exists an effective divisor $E$ such that either
\[
(2H + 2C) \cdot E = 1, E^2 = 0 \quad \text{or} \quad (2H + 2C) \cdot E = 0, E^2 = -1.
\]
The first case is clearly not possible. In the second case, $H \cdot E = 0$, and $C \cdot E = 0$. So $E$ is a component of one of the fibers of $\pi$ which is contracted by $f$ and which is a $(-1)$-curve. This contradicts the assumption that $(S, f)$ is minimal. Thus $(2H + 2C + K)^2 \geq 0$. Also, since $H^1(S, f^*O_X(-1)) = 0$, $H \cdot (H + K) = 2\chi(f^*O_X(-1)) - 2 \geq -2$, so we can write

$$\chi(N \otimes IC) = \frac{2n + 2 - d^2 - d}{2} H^2 - (n - d - 15)(e - 1) - 2 - 2(2H + 2C + K)^2 - \frac{n - d - 15}{2} (H \cdot (H + K) + 2)
$$

$$\leq \frac{2n + 2 - d^2 - d}{2} H^2 - (n - d - 15)(e - 1) - 2,$$

and therefore $\chi(N \otimes IC)$ is negative when $d^2 + d \geq 2n + 2$ and $n \geq 30$.

The Leray spectral sequence gives a short exact sequence

$$0 \to H^1(P^1, \pi_*(N \otimes IC)) \to H^1(S, N \otimes IC) \to H^1(P^1, R^1\pi_*(N \otimes IC)) \to 0,$$

and by our assumption on $S$ and $f$, $H^1(P^1, \pi_*(N \otimes IC)) = 0$. If we could choose $S$ such that $H^0(P^1, R^1\pi_*(N \otimes IC)) = 0$, then we could conclude that $\chi(N \otimes IC) \geq 0$ and hence $R_\pi(X)$ could not be uniruled for $d^2 + d \geq 2n + 2$ and $n \geq 30$.

We cannot show that for a general $X$, a minimal pair $(S, f)$ as in Proposition 2.1 can be chosen so that $H^0(P^1, R^1\pi_*(N \otimes IC)) = 0$. However, we prove that if $X$ is general and $(S, f)$ is minimal, then for every $t \geq 1$,

$$H^0(P^1, R^1\pi_*(N \otimes IC \otimes f^*O_X(t))) = 0.$$

We also show that if $t \geq 0$ and $f(C)$ is $t$-normal, then

$$H^1(P^1, \pi_*(N \otimes IC \otimes f^*O_X(t))) = 0.$$

These imply that $\chi(N \otimes IC \otimes f^*O_X(t))$ is non-negative when $X$ is general and $f(C)$ is $t$-normal. To finish the proof of Theorem 1.4, we compute $\chi(N \otimes IC \otimes f^*O_X(t))$ directly and show that it is negative when the inequality in the statement of the theorem holds.

**Proof of Theorem 1.4.** Let $X$ be a general hypersurface of degree $d$ in $P^n$. If $R_\pi(X)$ is uniruled, then there are $S$ and $f$ as in Proposition 2.1. Assume the pair $(S, f)$ is minimal. Let $N$ be the normal sheaf of $f$, and let $C$ be a general fiber of $\pi$. Then $H^0(S, N) \to H^0(C, N|_C)$ is surjective. The restriction map $H^0(S, f^*O_X(m)) \to H^0(C, f^*O_X(m)|_C)$ is also surjective since $f(C)$ is $m$-normal, so the restriction map $H^0(S, N \otimes f^*O_X(m)) \to H^0(C, N \otimes f^*O_X(m)|_C)$ is surjective as well. Therefore,

$$H^1(P^1, \pi_*(N \otimes f^*O_X(m) \otimes IC)) = 0.$$

Now let $C$ be an arbitrary fiber of $\pi$, and let $C^0$ be an irreducible component of $C$. Then by Proposition 5.2, $f^*(T_X(t))|_{C^0}$ is globally generated for every $t \geq 1$, and hence $N \otimes f^*O_X(t)|_{C^0}$ is globally generated too. So Lemma 5.1 shows that for every $t \geq 1$

$$H^0(P^1, R^1\pi_*(N \otimes f^*O_X(t) \otimes IC)) = 0.$$
By the Leray spectral sequence,

\[
H^1(S, N \otimes f^* \mathcal{O}_X(m) \otimes I_C) = H^1(P^1, \pi_*(N \otimes f^* \mathcal{O}_X(m) \otimes I_C)) \\
\quad \oplus H^0(P^1, R^1\pi_*(N \otimes f^* \mathcal{O}_X(m) \otimes I_C)) = 0,
\]

and therefore, \(\chi(N \otimes f^* \mathcal{O}_X(m) \otimes I_C) \geq 0\). We next compute \(\chi(N \otimes f^* \mathcal{O}_X(m) \otimes I_C)\). For an integer \(t \geq 0\), set \(a_t = \chi(N \otimes I_C \otimes f^* \mathcal{O}_X(t))\).

We have

\[
a_t = \chi(N \otimes I_C) + \frac{2t(n + 1 - d) + t^2(n - 3)}{2} H^2 - \frac{t(n - 5)}{2} H \cdot K - t(n - 3)e.
\]

So

\[
a_t = \frac{b_t}{2} H^2 + \frac{c_t}{2} H \cdot K + 2K^2 + d_t,
\]

where

\[
b_t = (n + 1 - d^2) + 2t(n + 1 - d) + t^2(n - 3),
\]

\[
c_t = -(n + 1 - d) - t(n - 5),
\]

and

\[
d_t = -t(n - 3)e - (n + 1 - d)e + 14.
\]

A computation similar to the computation in the beginning of this section shows that

\[
a_t = \frac{b_t - c_t}{2} H^2 - 2(2H + 2C + K)^2 + \frac{c_t + 16}{2} (H \cdot (H + K) + 2) + (d_t - c_t - 32 + 16e)
\]

\[
\leq \frac{b_t - c_t}{2} H^2 + (d_t - c_t - 32 + 16e).
\]

Since

\[
d_t - c_t - 32 + 16e = -(e - 1)(n - 15 - d + t(n - 3)) - 2t - 2,
\]

and since \(n - 15 - d + t(n - 3) \geq 2n - d - 18 \geq 0\) for \(t \geq 1\) and \(n \geq 12\), we get

\[
a_t < \frac{b_t - c_t}{2} H^2.
\]

When \(d^2 + (2t + 1)d \geq (t + 1)(t + 2)n + 2\), \(b_t < c_t\), and so \(a_t < 0\). If we let \(t = m\), we get the desired result.

\[\square\]

**Lemma 5.1.** If \(E\) is a locally free sheaf on \(S\) such that for every irreducible component \(C^0\) of a fiber of \(\pi\), \(E|_{C^0}\) is globally generated, then \(R^1\pi_* E = 0\).
**Proof.** By cohomology and base change [7, Theorem III.12.11], it suffices to prove that for every fiber $C$ of $\pi$, $H^1(C, E|_C) = 0$. We first show that if $l$ is the number of irreducible components of $C$ counted with multiplicity, then we can write $C = C_1 + \cdots + C_l$ such that each $C_i$ is an irreducible component of $C$ and for every $1 \leq i \leq l - 1$, $(C_1 + \cdots + C_i) \cdot C_{i+1} \leq 1$. This is proven by induction on $l$. If $l = 1$, there is nothing to prove. Otherwise, there is at least one component $C^0$ of $C$ which can be contracted. Let $r$ be the multiplicity of $C^0$ in $C$. Blowing down $C^0$, we get a rational surface $S'$ over $\mathbb{P}^1$. Denote by $C'$ the blow-down of $C$. Then by the induction hypothesis, we can write

$$C' = C'_1 + \cdots + C'_{l-r}$$

such that $(C'_1 + \cdots + C'_i) \cdot C'_{i+1} \leq 1$ for every $1 \leq i \leq l - r - 1$. Let $C_i$ be the proper transform of $C'_i$. Then if in the above sum we replace $C'_i$ by $C_i$ when $C_i$ does not intersect $C^0$ and by $C_i + C^0$ when $C_i$ intersects $C^0$, we get the desired result for $C$.

Since $E|_{C_{i+1}}$ is globally generated, it follows that

$$H^1(C_{i+1}, E(-C_1 - \cdots - C_i)|_{C_{i+1}}) = 0 \text{ for every } 0 \leq i \leq l - 1.$$

On the other hand, for every $0 \leq i \leq l - 2$, we have a short exact sequence of $\mathcal{O}_S$-modules

$$0 \to E(-C_1 - \cdots - C_{i+1})|_{C_{i+2} + \cdots + C_l} \to E(-C_1 - \cdots - C_{i})|_{C_{i+1} + \cdots + C_l} \to E(-C_1 - \cdots - C_i)|_{C_{i+1}} \to 0.$$

So a decreasing induction on $i$ shows that for every $0 \leq i \leq l - 2$, $H^1(S, E(-C_1 - \cdots - C_i)|_{C_{i+1} + \cdots + C_l}) = 0$. Letting $i = 0$, the statement follows. \qed

**Proposition 5.2.** Let $X \subset \mathbb{P}^n$ be a general hypersurface of degree $d$.

(i) For any morphism $h : \mathbb{P}^1 \to X$, $h^*(T_X(1))$ is globally generated.

(ii) If $C$ is a smooth, rational, $d$-normal curve on $X$, then $H^1(C, T_X|_C) = 0$.

**Proof.** (i) This follows from [13, Proposition 1.1]. We give a proof here for the sake of completeness. Consider the short exact sequence

$$0 \to h^*T_X \to h^*T_{\mathbb{P}^n} \to h^*\mathcal{O}_X(d) \to 0.$$

Since $X$ is general, the image of the pull-back map $H^0(X, \mathcal{O}_X(d)) \to H^0(\mathbb{P}^1, h^*\mathcal{O}_X(d))$ is contained in the image of the map $H^0(\mathbb{P}^1, h^*T_{\mathbb{P}^n}) \to H^0(\mathbb{P}^1, h^*\mathcal{O}_X(d))$. Choose a homogeneous coordinate system for $\mathbb{P}^n$. Let $p$ be a point in $\mathbb{P}^1$, and without loss of generality assume that $h(p) = (1 : 0 : \cdots : 0)$. We show that for any $r \in h^*(T_X(1))|_p$, there is $\tilde{r} \in H^0(\mathbb{P}^1, h^*(T_X(1)))$ such that $\tilde{r}|_p = r$.

Consider the exact sequence

$$0 \to H^0(\mathbb{P}^1, h^*T_X(1)) \to H^0(\mathbb{P}^1, h^*T_{\mathbb{P}^n}(1)) \to H^0(\mathbb{P}^1, h^*\mathcal{O}_X(d + 1)).$$

Denote by $s$ the image of $r$ in $h^*(T_{\mathbb{P}^n}(1))|_p$. There exists $S \in H^0(\mathbb{P}^n, T_{\mathbb{P}^n}(1))$ such that the restriction of $s := h^*(S)$ to $p$ is $s$. Denote by $T$ the image of $S$ in $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d + 1))$, and let $\tilde{t} = h^*(T)$. Then $T$ is a form of degree $d + 1$ on $\mathbb{P}^n$, and since $t|_p = 0$, we can write

$$T = x_1G_1 + \cdots + x_nG_n,$$
where the $G_i$ are forms of degree $d$. Our assumption implies that for every $1 \leq i \leq n$, there is $	ilde{s}_i \in H^0(\mathbb{P}^1, h^*T_{\mathbb{P}^n})$ such that $\phi(\tilde{s}_i) = h^*G_i$. Then

$$\phi(\tilde{s} - h^*(x_1)\tilde{s}_1 - \cdots - h^*(x_n)\tilde{s}_n) = \tilde{\ell} - h^*(x_1G_1) - \cdots - h^*(x_nG_n) = 0,$$

and therefore, $\tilde{s} - h^*(x_1)\tilde{s}_1 - \cdots - h^*(x_n)\tilde{s}_n$ is the image of some $\tilde{r} \in H^0(\mathbb{P}^1, h^*(T_X(1)))$. Since $(\tilde{s} - h^*(x_1)\tilde{s}_1 - \cdots - h^*(x_n)\tilde{s}_n)|_p = \tilde{s}|_p = s$, we have $\tilde{r}|_p = r$.

(ii) There is a short exact sequence

$$0 \to T_X|_C \to T_{\mathbb{P}^n}|_C \to \mathcal{O}_C(d) \to 0.$$

The fact that $X$ is general implies that any section of $\mathcal{O}_C(d)$ which is the restriction of a section of $\mathcal{O}_{\mathbb{P}^n}(d)$ can be lifted to a section of $T_{\mathbb{P}^n}|_C$. Since the first cohomology group of $T_{\mathbb{P}^n}|_C$ vanishes, the result follows.

Although for every $e$ and $n$ with $e \geq n + 1 \geq 4$, there are smooth non-degenerate rational curves of degree $e$ in $\mathbb{P}^n$ which are not $(e-n)$-normal [5, Theorem 3.1], a general smooth rational curve of degree $e$ in a general hypersurface of degree $d$ has possibly much smaller normality: if a maximal-rank type conjecture holds for rational curves contained in general hypersurfaces (at least when $d < \frac{n+2}{2}$), then it follows that if $c$ is the smallest positive number such that \[
\binom{n+c}{n} - \binom{n+c-d}{n} \geq ce + 1, \]
then a general smooth rational curve of degree $e$ in a general hypersurface of degree $d$ in $\mathbb{P}^n$ is $c$-normal.

\section*{References}


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