HW8, Problem 5. Let $C = V(x^2yz + xy^3 - x^4 - y^4)$. (There was a typo in the statement and the second term in $x^2yz + xy^3 - x^4 - y^4$ was written as $xy^2$.) Since the intersection of $C$ and $D$ consists of only one point: $(0 : 0 : 1)$, the multiplicity has to be $1 \times 4 = 4$ by Bezout’s theorem.

HW8, Problem 7. Let $L$ be the common tangent line to $C$ and $D$ at $p$. Note that after a linear change of coordinates, we can assume that $p = (1 : 0 : 0)$, $L = V(z)$, and $q = (0 : 0 : 1)$ is not on $L$, $C$, $D$, or any of the lines joining 2 points of intersection of $C$ and $D$. (to do so, first pick a point $q$ with the given property, and pick a point $r$ on $L$ other than $p$. Then the three points $p, q$, and $r$ give 3 linearly independent vectors in $\mathbb{C}^3$, and therefore, we can find an invertible matrix sending $q$ to $(0 : 0 : 1)$, $p$ to $(1 : 0 : 0)$, and $r$ to $(0 : 1 : 0)$. Then $L$ is sent to the unique line through $(1 : 0 : 0)$ and $(0 : 1 : 0)$ which is $V(z)$.)

Now we write

\[ f = a_n z^n + \cdots + a_1 z + a_0 \]

and

\[ g = b_m z^m + \cdots + b_1 z + b_0 \]

where $a_i$ is homogeneous of degree $n-i$ in $x, y$, and $b_i$ is homogeneous of degree $m-i$ in $x, y$. Form the matrix $A$ where \( \text{det} A = \text{Res}(f,g,z) \). The only non-zero elements in the last column of $A$ are $A_{m,n+m} = a_0$ and $A_{n+m,n+m} = b_0$. Since $C$ and $D$ pass through $(1 : 0 : 0)$, we have $a_0$ and $b_0$ are multiples of $y$.

Since the tangent line to $C$ at $p$ is given by

\[ f_x(1,0,0)x + f_y(1,0,0)y + f_z(1,0,0)z = 0 \]

and since we are assuming the tangent line is given by $z = 0$, we conclude, $f_y(1,0,0) = 0$. This implies that the coefficient of $y x^{n-1}$ in $f$ is zero (since it is the only possible term which could contribute to $f_y(1,0,0)$). This shows that $a_0$ is a multiple of $y^2$. Similarly $b_0$ is a multiple of $y^2$. Therefore, when we expand the determinant of $A$
with respect to the last column, we see that the determinant is a sum of polynomials which are multiples of $y^2$, and therefore the multiplicity at $p$ is at least 2.

HW 9, Question 2. It follows from what we proved in class that the points of inflection of a curve $C$ in $\mathbb{P}^2$ are the points $p$ of $C$ for which the determinant of the following symmetric matrix is zero:

$$A = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix}$$

The determinant of $A$ is a homogeneous polynomial $g$ of degree $3(d-2)$ and therefore $D := \mathbf{V}(g)$ is of degree $3(d-2)$. Let $C = \mathbf{V}(f)$ where $f$ is a homogeneous polynomial of degree $d$, then by Bezout’s theorem, the number of points in $C \cap D$ is at most $3d(d-2)$.

HW 9, Question 3. By definition, the degree of a curve $C$ in $\mathbb{P}^3$, is the maximum number of points in the intersection of $C$ and a hyperplane in $\mathbb{P}^3$ (assuming $C$ and the hyperplane intersect at a finite number of points.). A hyperplane $H$ in $\mathbb{P}^3$ is given by equation $ax_0 + bx_1 + cx_2 + dx_3 = 0$ where $a, b, c, d$ are constant. So the number of points in the intersection of $C$ and $H$ is exactly the number of solutions of the degree 3 homogeneous polynomial

$$a t^3 + b t^2 s + c ts^2 + d c^3$$

in $\mathbb{P}^1$, and the maximum of the number of solutions is 3. So $C$ has degree 3.

HW9, Question 5. Set $f = y^2z - x^3 - axz^2 - bx^3$. We know that the inverse of a point with coordinates $(x : y : z)$ is the point with coordinates $(x : -y : z)$. To find the coordinates of $2p$, we look at the line tangent to $C$ at $p = (x_0 : y_0 : 1)$, find the 3rd point of intersection of the line with $C$, and $2p$ will be the inverse of that point. We should assume that $y_0 \neq 0$ for this problem. The equation of the tangent line at $p$ is given by

$$\alpha x + \beta y + \gamma z = 0$$

where $\alpha = f_x(x_0, y_0, 1) = -3x_0^2 - a$, $\beta = f_y(x_0, y_0, 1) = 2y_0 \neq 0$, $\gamma = f_z(x_0, y_0, 1)$. The points of intersection of the tangent line with $C$ is the common zero locus of the above linear polynomial and $f$. From the linear equation we get

$$y = -\frac{\alpha}{\beta} x - \frac{\gamma}{\beta} z$$
and from \( f \) we get

\[
y^2z = (-\frac{\alpha}{\beta}x - \frac{\gamma}{\beta}z)^2z = x^3 + axz^2 + bz^3.
\]

So the coordinates \((x : y : z)\) of the points of intersection satisfy

\[
x^3 - \frac{\alpha^2}{\beta^2}x^2z + (a - \frac{2\alpha\gamma}{\beta^2})xz^2 + (b - \frac{\gamma^2}{\beta^2})z^3 = 0.
\]

Note that \( z \) cannot be zero, since then \( x \) would be zero too, and hence \( y = \frac{\alpha}{\beta}x - \frac{\gamma}{\beta}z = 0 \).

So (after dividing the homogeneous coordinates by \( z \)), we can assume \( z = 1 \) for all the common roots, and \( x \) satisfies

\[
x^3 - \frac{\alpha^2}{\beta^2}x^2 + (a - \frac{2\alpha\gamma}{\beta^2})x + (b - \frac{\gamma^2}{\beta^2}) = 0.
\]

The sum of roots of the above polynomial is negative the coefficient of \( x^2 \) which is \( \frac{\alpha^2}{\beta^2} \).

Since \( x_0 \) is a root of multiplicity 2, so the \( x \)-coordinate of the the other root should be

\[
\frac{\alpha^2}{\beta^2} - 2x_0 = \left(\frac{3x_0^2 + a}{2y_0}\right)^2 - 2x_0.
\]