Surface Integrals

Here is a summary of how to find surface integrals of functions and vector fields.

If we have a surface S, then we can integrate a function or a vector field over S. To integrate a function or a vector field over S, we first parametrize S. Since S is two-dimensional, we always need two parameters u and v, and we should express x, y and z in terms of our parameters.

$$\mathbf{r}(u,v) =< x(u,v), y(u,v), z(u,v) > .$$

We then compute $\mathbf{r}_u \times \mathbf{r}_v$.

It might be tricky to find the correct parametrization, but here are a few general rules:

1. Use spherical coordinates only if the surface is a part a sphere.

2. If S is a part of a cylinder, for example $x^2 + y^2 = 1$, then use θ and z to parametrize S. So S is parametrized by

$$\mathbf{r}(\theta, z) =< \cos(\theta), \sin(\theta), z > .$$

In this case, $\mathbf{r}_\theta \times \mathbf{r}_z =< \cos(\theta), \sin(\theta), 0 >$.

3. If S is a part of a plane with equation $ax + by + cz = d$, then use x and y as parameters

$$\mathbf{r}(x,y) =< x, y, \frac{d}{c} - \frac{a}{c}x - \frac{b}{c}y > .$$

In this case, $\mathbf{r}_x \times \mathbf{r}_y =< \frac{a}{c}, \frac{b}{c}, 1 >$.

4. If S is a part of a graph of a function $z = f(x,y)$, we can use x and y as parameters, so

$$\mathbf{r}(x,y) =< x, y, f(x,y) > .$$

In this case, $\mathbf{r}_x \times \mathbf{r}_y =< -f_x, -f_y, 1 >$. If for example, S is the paraboloid $z = x^2 + y^2$, and if we choose x and y as our parameteres, $\mathbf{r}_x \times \mathbf{r}_y =< -2x, -2y, 1 >$, and if S is the upper half of the cone $z^2 = x^2 + y^2$, so

$z = \sqrt{x^2+y^2}$, we can choose x and y as our parameters and $\mathbf{r}_x \times \mathbf{r}_y =< \frac{-x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}}, 1 >$,

Of course, in each of these case, you should also find the region where the parameters come from. And you may need to change these sometimes a little bit. For example if we have a surface given by an equation like $y = g(x, z)$, then it is easier to use x and z as parameters.

Once you have the parametrization of S, say

$$\mathbf{r}(u, v) =< x(u, v), y(u, v), z(u, v) >,$$

then

- You find the surface area of S by

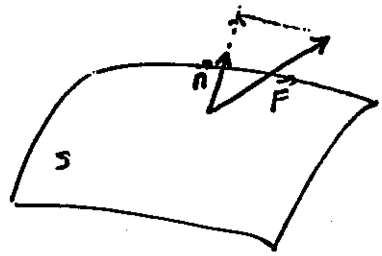
$$\text{area} = \iint_R |\mathbf{r}_u \times \mathbf{r}_{\mathbf{v}}|\, dA.$$

- If $f(x, y, z)$ is a function, then we can take the integral of f over S. This is called a surface integral, and we have had two different notations: in the book $\iint_S f\, d\sigma$ and in webwork $\iint_S f\, dS$.

$$\text{inegral of } f \text{ over } S = \iint_R f\, |\mathbf{r}_u \times \mathbf{r}_v|\, dA.$$

  So the area is equal to the surface integral of the constant function 1.

- If we have a vector field $\mathbf{F}$, we can also take the integral of $\mathbf{F}$ over S. But to do so, we need to first fix an orientation of S say $\mathbf{n}$. If we look at $f = \mathbf{F} \cdot \mathbf{n}$, then we get a function whose value at every point is the scalar component of $\mathbf{F}$ in the direction of $\mathbf{n}$.



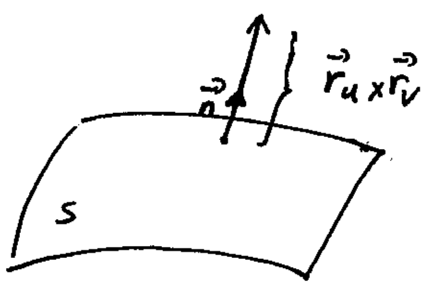
scalar component of the $\operatorname{proj}_{\vec{n}} \vec{F} = \dfrac{\vec{F}\cdot\vec{n}}{|\vec{n}|} = \vec{F}\cdot\vec{n}$

the integral of $\mathbf{F}$ with respect to $\mathbf{n}$ is defined to be the integral of the function f. This is sometimes called the flux of $\mathbf{F}$ with respect to $\mathbf{n}$, and sometimes simply the surface integral of $\mathbf{F}$ with respect to $\mathbf{n}$.

There are two notations for the integral of a vector field over a surface: in the book $\int_S \mathbf{F}\cdot\mathbf{n}\,d\sigma$, and in the webwork $\int_S \mathbf{F}\cdot d\mathbf{S}$.

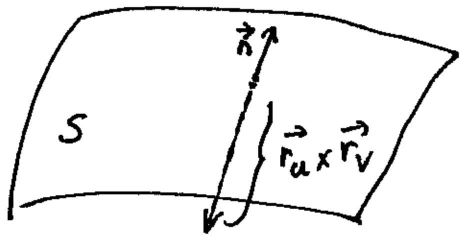
How do you find the integral of $\mathbf{F}$? You look at a parametrization, you look at $\mathbf{r}_u \times \mathbf{r}_v$. This vector is always normal to the surface. if it was in the same direction as the orientation of S, then $\mathbf{n} = \frac{\mathbf{r}_u\times\mathbf{r}_v}{|\mathbf{r}_u\times\mathbf{r}_v|}$, so

$$\begin{aligned}\text{integral of } \mathbf{F} \text{ with respect to } \mathbf{n} &= \iint_R \mathbf{F}\cdot\left(\frac{\mathbf{r}_u\times\mathbf{r}_v}{|\mathbf{r}_u\times\mathbf{r}_v|}\right)|\mathbf{r}_u\times\mathbf{r}_v|\,dA \\ &= \iint_R \mathbf{F}\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,dA\end{aligned}$$



If $\mathbf{r}_u \times \mathbf{r}_v$ is in the opposite direction as the orientation of S, then $\mathbf{n} = -\frac{\mathbf{r}_u\times\mathbf{r}_v}{|\mathbf{r}_u\times\mathbf{r}_v|}$, so

$$\text{integral of } \mathbf{F} \text{ with respect to } \mathbf{n} = -\iint_R \mathbf{F}\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,dA.$$



Finally, Stokes' theorem says that is S is a surface with an orientation $\mathbf{n}$, and C is the boundary of S which is oriented positively with respect to $\mathbf{n}$, then for a vector field $\mathbf{F}$, the line integral of $\mathbf{F}$ on the boundary is equal to the surface integral of curl $\mathbf{F}$ over S, with respect to $\vec{n}$.