1.(1 pt) 1 3
If $\{a_n\}$ is a sequence of real numbers, the first difference ∇a_n is the sequence given by

$$\nabla a_n = \begin{cases} a_n - a_{n-1} \text{ if } n > 1 \\ 0 \text{ if } n = 1. \end{cases}$$

For example if $\{a_n\}$ is the sequence $1,3,5,7,9,\ldots$ then ∇a_n is the sequence $0,2,2,2,2,\ldots$. Notice the sequence ∇a_n always starts with 0 and the subsequent entries keep track of the differences in the original sequence $\{a_n\}$. Fill in the blanks below:

$a_n : 1,3,9,27,81,\ldots$
$\nabla a_n : 0,2,6,18,54,\ldots$
$b_n : 1,2,5,10,17,26,\ldots$
∇b_n : 0,____ ,____ ,____ , ____ , ____
$c_n : 1,2,5,14,41,122,\ldots$
∇c_n : 0, ____ ,____ ,____ , ____ , ____

Similarly one defines $\nabla^2 a_n$ by:

$$\nabla^2 a_n = \begin{cases} \nabla a_n - \nabla a_{n-1} \text{ if } n > 1 \\ 0 \text{ if } n = 1 \end{cases}$$

So for example:
$a_n : 1,2,3,5,8,13,21,\ldots$
$\nabla a_n : 0,1,1,2,3,5,8,\ldots$
$\nabla^2 a_n : 0,1,0,1,1,2,3,\ldots$

Fill in the following blanks:
$b_n : 0,1,3,6,10,15,21,\ldots$, (Note $b_n = C(n,2)$).
∇b_n : 0, ____ ,____ ,____ ,____ , ____
$\nabla^2 b_n$: 0, ____ ,____ ,____ ,____ , ____

2.(1 pt)
Decide if each of the following recurrence relations is a linear homogeneous recurrence with constant coefficients (lhcc). Answer "Y" for yes and "N" for no.

__ 1. $a_n = 3a_{n-1} + 4a_{n-2} + 5a_{n-3}$
__ 2. $a_n = 2na_{n-1} + a_{n-2}$
__ 3. $a_n = 3$
__ 4. $a_n = a_{n-1}^2$
__ 5. $a_n = a_{n-2}$
__ 6. $a_n = a_{n-1} + a_{n-4}$

Find the degree of the following lhcc recurrences:

__ 1. $a_n = 2a_{n-1} + 2a_{n-6}$
__ 2. $a_n = a_{n-1} + a_{n-2} + a_{n-3}$
__ 3. $a_n = 7a_{n-1} + 9a_{n-5}$
__ 4. $a_n = 5a_{n-1}$

3.(1 pt) We will find the solution to the following lhcc recurrence:
$a_n = 3a_{n-1} + 4a_{n-2}$ for $n \geq 2$ with initial conditions $a_0 = 4, a_1 = 5$.

The first step in any problem like this is to find the characteristic equation by trying a solution of the "geometric" format $a_n = r^n$. (We assume also $r \neq 0$). In this case we get:

$$r^n = 3r^{n-1} + 4r^{n-2}.$$

Since we are assuming $r \neq 0$ we can divide by the smallest power of r, i.e., r^{n-2} to get the characteristic equation:

$$r^2 = 3r + 4.$$

(Notice since our lhcc recurrence was degree 2, the characteristic equation is degree 2.)

Find the two roots of the characteristic equation r_1 and r_2. When entering your answers use $r_1 \leq r_2$:
$r_1 =$ ________ , $r_2 =$ ________

Since the roots are distinct, the general theory (Theorem 1 in section 5.2 of Rosen) tells us that the general solution to our lhcc recurrence looks like:

$$a_n = \alpha_1(r_1)^n + \alpha_2(r_2)^n$$

for suitable constants α_1, α_2.

To find the values of these constants we have to use the initial conditions $a_0 = 4, a_1 = 5$. These yield by using n=0 and n=1 in the formula above:

$$4 = \alpha_1(r_1)^0 + \alpha_2(r_2)^0$$

and

$$5 = \alpha_1(r_1)^1 + \alpha_2(r_2)^1$$

By plugging in your previously found numerical values for r_1 and r_2 and doing some algebra, find α_1, α_2:
[Be careful to note that $(-x)^n \neq -(x^n)$ when n is even, for example $(-3)^2 \neq -(3^2)$.]
$\alpha_1 =$ ________
$\alpha_2 =$ ________

Note the final solution of the recurrence is:

$$a_n = \alpha_1(r_1)^n + \alpha_2(r_2)^n$$

where the numbers r_i, α_i have been found by your work. This gives an explicit numerical formula in terms of n for the a_n.

4.(1 pt) Find the solution to the following lhcc recurrence: $a_n = 3a_{n-1} + 18a_{n-2}$ for $n \geq 2$ with initial conditions $a_0 = 3, a_1 = 5$.

The solution is of the form:

$$a_n = \alpha_1(r_1)^n + \alpha_2(r_2)^n$$

for suitable constants $\alpha_1, \alpha_2, r_1, r_2$ with $r_1 \leq r_2$. Find these constants.
$r_1 =$ ____ $r_2 =$ ____ $\alpha_1 =$ ____ $\alpha_2 =$ ____

5.(1 pt) Find the solution to the following lhcc recurrence: $a_n = 9a_{n-2}$ for $n \geq 2$ with initial conditions $a_0 = 2, a_1 = 2$.

The solution is of the form:

$$a_n = \alpha_1(r_1)^n + \alpha_2(r_2)^n$$

for suitable constants $\alpha_1, \alpha_2, r_1, r_2$ with $r_1 \leq r_2$. Find these constants.

$r_1 =$ ______ $r_2 =$ ______ $\alpha_1 =$ ______ $\alpha_2 =$ ______

6.(1 pt) We will find the solution to the following lhcc recurrence:

$a_n = -2a_{n-1} - 1a_{n-2}$ for $n \geq 2$ with initial conditions $a_0 = 1, a_1 = 5$.

The first step as usual is to find the characteristic equation by trying a solution of the "geometric" format $a_n = r^n$. (We assume also $r \neq 0$). In this case we get:

$$r^n = -2r^{n-1} - 1r^{n-2}.$$

Since we are assuming $r \neq 0$ we can divide by the smallest power of r, i.e., r^{n-2} to get the characteristic equation:

$$r^2 = -2r - 1.$$

(Notice since our lhcc recurrence was degree 2, the characteristic equation is degree 2.)

This characteristic equation has a single root r. (We say the root has multiplicity 2). Find r.

$r =$ ______

Since the root is repeated, the general theory (Theorem 2 in section 5.2 of Rosen) tells us that the general solution to our lhcc recurrence looks like:

$$a_n = \alpha_1(r)^n + \alpha_2 n(r)^n$$

for suitable constants α_1, α_2.

To find the values of these constants we have to use the initial conditions $a_0 = 1, a_1 = 5$. These yield by using n=0 and n=1 in the formula above:

$$1 = \alpha_1(r)^0 + \alpha_2 0(r)^0$$

and

$$5 = \alpha_1(r)^1 + \alpha_2 1(r)^1$$

By plugging in your previously found numerical value for r and doing some algebra, find α_1, α_2:

$\alpha_1 =$ ______

$\alpha_2 =$ ______

Note the final solution of the recurrence is:

$$a_n = \alpha_1(r)^n + \alpha_2 n(r)^n$$

where the numbers r, α_i have been found by your work. This gives an explicit numerical formula in terms of n for the a_n.

7.(1 pt) We will find the solution to the following lhcc recurrence:

$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ for $n \geq 3$ with initial conditions $a_0 = 4, a_1 = 6, a_2 = 6$.

The first step as usual is to find the characteristic equation by trying a solution of the "geometric" format $a_n = r^n$. (We assume also $r \neq 0$). In this case we get:

$$r^n = 6r^{n-1} - 11r^{n-2} + 6r^{n-3}.$$

Since we are assuming $r \neq 0$ we can divide by the smallest power of r, i.e., r^{n-3} to get the characteristic equation:

$$r^3 = 6r^2 - 11r + 6.$$

(Notice since our lhcc recurrence was degree 3, the characteristic equation is degree 3.)

Find the three roots of the characteristic equation r_1, r_2 and r_3. When entering your answers use $r_1 \leq r_2 \leq r_3$:

$r_1 =$ ______ , $r_2 =$ ______ , $r_3 =$ ______ .

Since the roots are distinct, the general theory (Theorem 3 in section 5.2 of Rosen) tells us that the general solution to our lhcc recurrence looks like:

$$a_n = \alpha_1(r_1)^n + \alpha_2(r_2)^n + \alpha_3(r_3)^n$$

for suitable constants $\alpha_1, \alpha_2, \alpha_3$.

To find the values of these constants we have to use the initial conditions $a_0 = 4, a_1 = 6, a_2 = 6$. These yield by using n=0,n=1 and n=2 in the formula above:

$$4 = \alpha_1(r_1)^0 + \alpha_2(r_2)^0 + \alpha_3(r_3)^0$$

and

$$6 = \alpha_1(r_1)^1 + \alpha_2(r_2)^1 + \alpha_3(r_3)^1$$

and

$$6 = \alpha_1(r_1)^2 + \alpha_2(r_2)^2 + \alpha_3(r_3)^2$$

By plugging in your previously found numerical values for r_1, r_2 and r_3 and doing some algebra, find $\alpha_1, \alpha_2, \alpha_3$:

Note: Ad hoc substitution should work to find the α_i but for those who know linear algebra, note the system of equations above can be written in matrix form as:

$$\begin{bmatrix} (r_1)^0 & (r_2)^0 & (r_3)^0 \\ (r_1)^1 & (r_2)^1 & (r_3)^1 \\ (r_1)^2 & (r_2)^2 & (r_3)^2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 6 \end{bmatrix}$$

$\alpha_1 =$ ______

$\alpha_2 =$ ______

$\alpha_3 =$ ______

Note the final solution of the recurrence is:

$$a_n = \alpha_1(r_1)^n + \alpha_2(r_2)^n + \alpha_3(r_3)^n$$

where the numbers r_i, α_i have been found by your work. This gives an explicit numerical formula in terms of n for the a_n.