1.(1 pt)
Consider the function $y = f(x)$ specified by the following table:

-1	0.707106781186547
-0.8	-0.827080574274562
-0.6	-0.612907053652977
-0.4	0.562083377852131
-0.2	0.999506560365732
0	0.707106781186547
0.2	0.278991106039229
0.4	0.0314107590781282
0.6	0.0314107590781283
0.8	0.278991106039229
1	0.707106781186547

(The first column contains x values, while the second column contains the corresponding y values.) Find a numerical approximation to the function

$$F(x) = \int_{0.2}^{x} f(t)\,dt$$

using the following variant of midpoint Riemann sums: instead of computing

$$f\left(\frac{x_1 + x_2}{2}\right)$$

(whose value is not given) compute

$$\frac{f(x_1) + f(x_2)}{2}$$

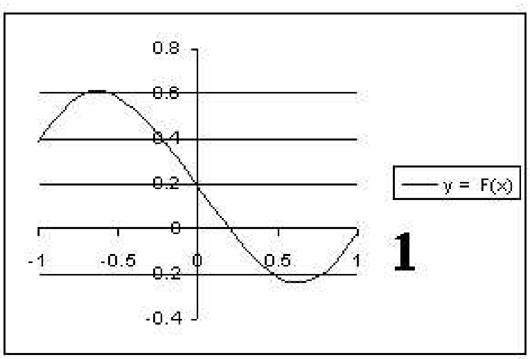
(x_1 and x_2 denote adjacent x values in the table.) Use a spreadsheet to do the calculation.

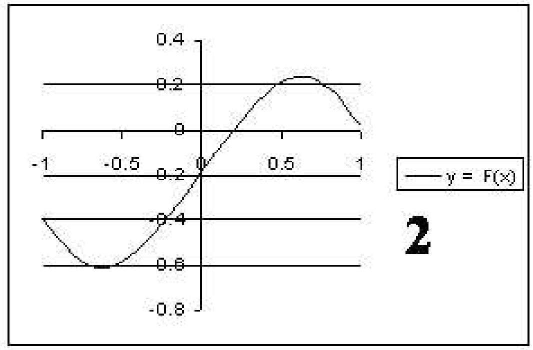
Then answer the following questions.

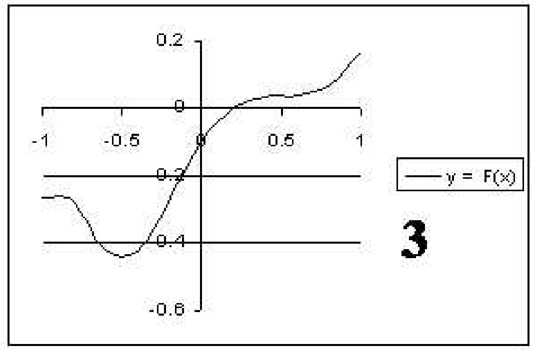
When $x = 0.8$ then $F(x) \approx$ ____________________
When $x = 0$ then $F(x) \approx$ ____________________
When $x = -0.6$ then $F(x) \approx$ ____________________
Look at the eight graphs below and choose the one which most closely resembles the graph of

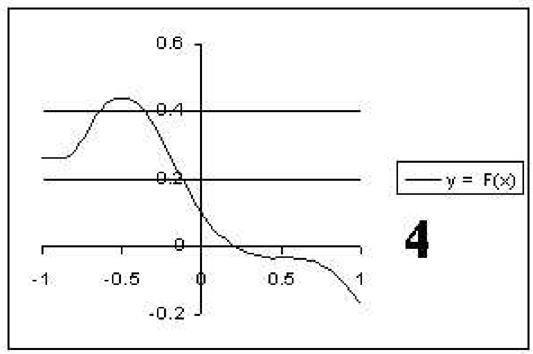
$$F(x) = \int_{0.2}^{x} f(t)\,dt$$

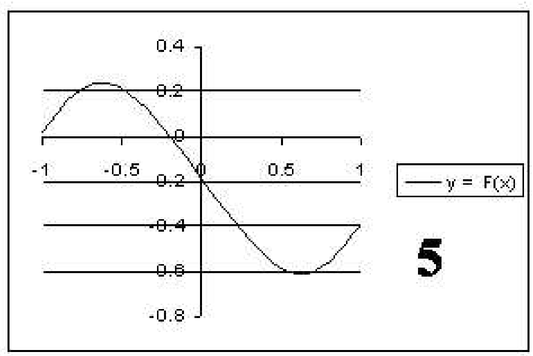
ANSWER = _____ . (Enter the label of the graph you think is right: 1, 2, 3, 4, 5, 6, 7 or 8.)

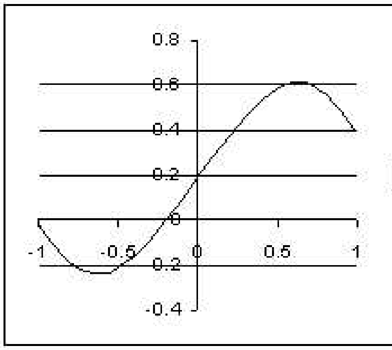












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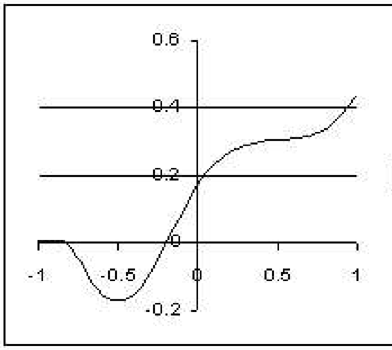

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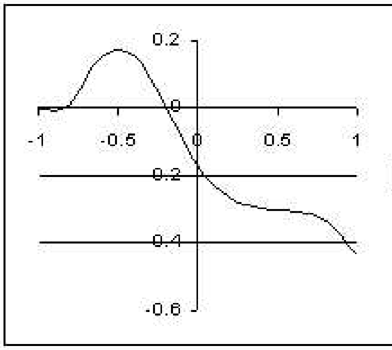


8

2.(1 pt)
Consider the function

$$F(x) = \int_0^x \sin(1.0975t^2)dt$$

Find a numerical approximation to this function using midpoint Riemann sums and plot it using a spreadsheet. Use equally spaced partitions of size $\Delta x = \pm 0.1$. Then numerically compute an approximation to the derivative $F'(x)$ and compare the resulting graph to that of $\sin(1.0975x^2)$.

To get credit for this problem you need to submit separate printouts (during lecture or office hours). of each of the following:

(1) The spreadsheet Tables 1, 2 and 3 with numerical values as indicated below. To conserve paper, just print the first two pages of each table.
(2) The spreadsheet Tables 1, 2 and 3 with spreadsheet formulas displayed instead of numbers. To conserve paper, just print the first two pages of each table.
(3) The graph of $y = F(x)$ using the spreadsheet chart facility.
(4) The graphs of $y = F'(x)$ and $y = \sin(1.0975x^2)$ plotted together, also using the spreadsheet chart facility.
(5) A printout of this page, after you have clicked the Submit button, indicating that you have answered the questions below correctly. (Don't bother submitting anything else without a WeBWorK printout showing you have at least computed the F(x) values correctly.)

The spreadsheet Tables 1 and 2 should have the following format:

- The A column should contain x values starting with $x = 0$ going up to $x = \pm 10$ in increments of the step size ± 0.1.
- The B column should contain the step size (interval width) Δx.
- The C column should contain the midpoint of the interval whose left endpoint is in column A and whose width is in column B.
- The D column should contain the value of $f(x) = \sin(1.0975x^2)$ at the midpoint in column C
- The E column should contain the (signed) area of the approximating rectangle over the interval whose left endpoint is in column A.
- The F column should contain an approximation to the value of $y = F(x)$ corresponding to the x value in column A. This approximation is obtained by summing up the areas of the approximating rectangles in column E in all the rows above the current one (EXCLUDING the current row).

You spreadsheet should consist of two separate tables, Table 1 using a step size of $\Delta x = 0.1$ (stepping forward), Table 2 using a step size of $\Delta x = -0.1$ (stepping backward).

For the plot chart of $y = F(x)$ follow these basic instructions, modifying as needed to suit your spreadsheet software.

(1) Select all the cells in your spreadsheet and "Copy"
(2) Open a new blank worksheet
(3) Click on the first cell and "Paste Special$\longrightarrow$ Values" into the new worksheet.
(4) Delete columns B, C, D, and E. The resulting spreadsheet will have x values in column A and corresponding $y = F(x)$ values in column B.
(5) Select all the cells in your new spreadsheet and "Sort" them in ascending order according to column A.
(6) Again select all the cells in your spreadsheet and, using the appropriate menu or task bar icon, create a chart plotting the selected cells
(7) When asked for the type of chart you want, select XY scatter chart with X values in column A. Then select the choice of chart which connects points by smooth curves, without dotting the points.
(8) The x values on your chart should range from -10 to 10.

(9) Save the spreadsheet from Step 5 above to start Table 3 for the second part of the problem.

The following questions are designed to tell you whether your methods are correct, and to debug your spreadsheets. Answer them by referring to Tables 1, 2 and 3 prepared according to the above instructions. You will need to submit a printout of the response from Webwork indicating that your answers are correct.

When $x = 1.1$ then $F(x) \approx$ ______
When $x = 2.2$ then $F(x) \approx$ ______
When $x = -0.8$ then $F(x) \approx$ ______
When $x = -1.7$ then $F(x) \approx$ ______

3.(1 pt)

Consider the function

$$F(x) = \int_0^x \sin(1.0975t^2)dt$$

of Part 1.

Now we are ready for the second part of the problem: the comparison between the graphs of $y = F'(x)$ and $y = \sin(1.0975x^2)$. Recall that

$$F'(x) = \lim_{h\to 0} \frac{F(x+h)-F(x)}{h}$$

Since we have only computed values of $F(x)$ in steps of size 0.1, the smallest value of h we can use is $h = 0.1$ giving us an approximation:

$$F'(x) \approx \frac{F(x+0.1)-F(x)}{0.1}$$

Start with the spreadsheet you saved from the previous part, whose first column contains x values from -10 to 10 going up in increments of 0.1. The second column contains the corresponding values $F(x)$. Now start a third column computing approximations to $F'(x)$ using the above approximation.

The third column should end one row early, corresponding to $x = 9.9$, since computing cell C201 $= F'(10)$ by this method, would require knowing $F(10+0.1) = F(10.1)$, which we haven't computed.

Finally fill in the fourth column by computing the values of $\sin(1.0975x^2)$ corresponding to the x values in the first column. This completes the Table 3 spreadsheet, which you are supposed to hand in (in two forms, one displaying numbers, the second displaying formulas).

You will notice that the values in the third column $F'(x)$ roughly match those in the fourth column $\sin(1.0975x^2)$ for x values in the range $-4 \le x \le 4$, but tend to diverge from each other outside this interval. Select this region $-4 \le x \le 4$ in your table and paste it into a new spreadsheet using the "Paste Special⟶ Values" command. Then delete the second column (containing $F(x)$ values), leaving a spreadsheet containing x values (from -4 to 4) in the first colum, corresponding $F'(x)$ values in the second column, and corresponding $\sin(1.0975x^2)$ values in the third column. Select all the cells in this spreadsheet and, using the appropriate menu or task bar icon, create a chart plotting the selected cells fas in the first part of problem.

The following questions are designed to tell you whether your methods are correct, and to debug your spreadsheets. Answer them by referring to Tables 1, 2 and 3 prepared according to the above instructions. You will need to submit a printout of the response from Webwork indicating that your answers are correct.

When $x = 1.1$ then $F'(x) \approx$ ______
When $x = 2.2$ then $F'(x) \approx$ ______
When $x = -0.8$ then $F'(x) \approx$ ______
When $x = -1.7$ then $F'(x) \approx$ ______

4.(1 pt)

Consider the graph from Part 2 of the problem. The approximate graph of $y = F'(x)$ looks like the graph of $y = \sin(1.0975x^2)$ shifted ____ units right. (Use a minus sign if the graph is shifted left.)

CLUES: Compare the column containing approximate values of F'(x) in Table 3 with various columns in Tables 1 and 2 (Table 1 is easier). You should be able to find a close match. Then note that this column (in Table 1 or 2) is obtained by applying the function $\sin(1.0975x^2)$ not to the x-values (which are in column A) but to x values with a shift.

It is perfectly kosher to work on this project in groups, as long as each individual submits their own work for their own individual version of the problem. Do not hesitate to seek help from your lecturer or TA, if you don't understand the mathematics or if you have technical difficulties with the spreadsheet.

EXTRA BONUS PROBLEM: You can improve the accuracy of the above calculations (especially the second graph, also extending the range of x values to a bigger interval than $-4 \le x \le 4$) by using a smaller step size than 0.1. Redo the entire problem using a smaller step size and explaining what you think is going on. (Be sure to explain what you think the value of the shift will be for this smaller step size.) Hand in the results separately writing your name and BONUS PROJECT 1 on the cover sheet. (You still have to hand in the regular Project 1.) To conserve paper, print out just the first two pages of Tables 1, 2 and 3. If you do this correctly, you will get 10 more bonus points added to your course total, after the grading curve for the class has been constructed.

5.(1 pt) Consider the exponential equation

$$1.1^x = 2$$

Let N and $N+1$ be the two INTEGERS which bracket the solution (ie. $1.1^N < 2$ and $1.1^{N+1} > 2$). Then
$N =$ ______

Now consider the exponential equation

$$1.1^y = 3$$

Let N and $N+1$ be the two INTEGERS which bracket the solution (ie. $1.1^N < 3$ and $1.1^{N+1} > 3$). Then
$N =$ ______

Using the integers N as approximations to the actual solutions x and y of the exponential equations above we find that the exponential equation

$$2^z = 3$$

has approximate solution $z \approx \frac{m}{n}$ where
$m =$ __________

and $n =$ __________

To see how good an approximation $\frac{m}{n}$ is to the actual solution we compute
$2^{\frac{m}{n}} =$ __________
and check how close it is to 3.

This problem illustrates John Napier's (1550-1617) approach to solving exponential equations and how he came to discover natural logarithms. Note that computing INTEGER powers of 1.1 can be done easily by hand, as multiplication by 1.1 requires only shifting and adding.

HINT FOR PARTS 3 AND 4: $(1.1^a)^b = 1.1^{ab}$

6.(1 pt) This problem is a reprise of Problem 1 with 1.1 replaced everywhere by 1.001

Consider the exponential equation

$$1.001^x = 2$$

Let N and $N+1$ be the two INTEGERS which bracket the solution (ie. $1.001^N < 2$ and $1.001^{N+1} > 2$). Then
$N =$ __________

Now consider the exponential equation

$$1.001^y = 3$$

Let N and $N+1$ be the two INTEGERS which bracket the solution (ie. $1.001^N < 3$ and $1.001^{N+1} > 3$). Then
$N =$ __________

Using the integers N as approximations to the actual solutions x and y of the exponential equations above we find that the exponential equation

$$2^z = 3$$

has approximate solution $z \approx \frac{m}{n}$ where
$m =$ __________

and $n =$ __________

To see how good an approximation $\frac{m}{n}$ is to the actual solution we compute
$2^{\frac{m}{n}} =$ __________
and check how close it is to 3.

This problem illustrates John Napier's (1550-1617) approach to solving exponential equations and how he came to discover natural logarithms. Note that computing INTEGER powers of 1.001 can be done easily by hand (given enough free time), as multiplication by 1.001 requires only shifting and adding.

HINT FOR PARTS 3 AND 4: $(1.001^a)^b = 1.001^{ab}$

7.(1 pt) The solution x to the exponential equation

$$a^x = b$$

with a and b given numbers is denoted $x = \log_a b$. Thus in problems 1 and 2 you were computing $\text{Int}(\log_{1.1} 2)$, $\text{Int}(\log_{1.001} 2)$, $\text{Int}(\log_{1.1} 3)$ and $\text{Int}(\log_{1.001} 3)$ (where Int( ) denotes the integer part), as well as approximations to $\log_2 3$.

Continue your calculations from the first part of problem 2 and compare your results with those you get computing natural logarithms (using a calculator or spreadsheet).

$\text{Int}(\log_{1.001} 2) =$ __________
while $\ln 2 =$ __________
$\text{Int}(\log_{1.001} 3) =$ __________
while $\ln 3 =$ __________
$\text{Int}(\log_{1.001} 4) =$ __________
while $\ln 4 =$ __________
$\text{Int}(\log_{1.001} 5) =$ __________
while $\ln 5 =$ __________
$\text{Int}(\log_{1.001} 6) =$ __________
while $\ln 6 =$ __________
$\text{Int}(\log_{1.001} 7) =$ __________
while $\ln 7 =$ __________
$\text{Int}(\log_{1.001} 8) =$ __________
while $\ln 8 =$ __________
$\text{Int}(\log_{1.001} 9) =$ __________
while $\ln 9 =$ __________
$\text{Int}(\log_{1.001} 10) =$ __________
while $\ln 10 =$ __________

8.(1 pt) Referring back to the results of the previous problem we find that there is a nice relation between natural logarithms and $\text{Int}(\log_{1.001}(-))$ namely

$$\ln(x) \approx \frac{\text{Int}(\log_{1.001} x)}{n}$$

where $n =$ __________

Let's try to explain this relation. Use the same procedure as in the preceding two problems to find $\text{Int}(\log_{1.001}(e)$ where $e = 2.718281828459\ldots$ is the base of natural logarithms. We find
$\text{Int}(\log_{1.001}(e)) =$ __________

Now use the same procedure as you used in problem 2 (to solve $2^z = 3$) to find an approximate solution of the exponential equation

$$e^z = 7$$

You find that $z \approx \frac{m}{n}$ where
$m =$ __________
and
$n =$ __________
Now recall that by definition $z = \ln(7)$.

Let's now explore this a little further. Suppose that we replace 1.001 in our calculations by 1.0005. Then we find
$\text{Int}(\log_{1.0005}(e)) =$ __________
and
$\text{Int}(\log_{1.0005}(7)) =$ __________
which gives the approximate solution $z \approx \frac{m}{n}$ to $e^z = 7$ where
$m =$ __________
and
$n =$ __________
Generalizing this we get the following relation:

$$\ln(x) \approx \frac{\text{Int}(\log_{1.0005} x)}{n}$$

where $n =$ ______ .

Similarly we find that

$\text{Int}(\log_{1+1/3059}(e)) =$ ______

and that

$$\ln(x) \approx \frac{\text{Int}(\log_{1+1/3059} x)}{n}$$

where $n =$ ______ .

By the same method we find that

$$\ln(x) \approx \frac{\text{Int}(\log_{1+1/32564} x)}{n}$$

where $n =$ ______ .

The reason we get this pattern is because of the following limit formula

$$e = \lim_{n \to \infty} \text{\_\_}$$

which is usually taken to be the definition of the number e.

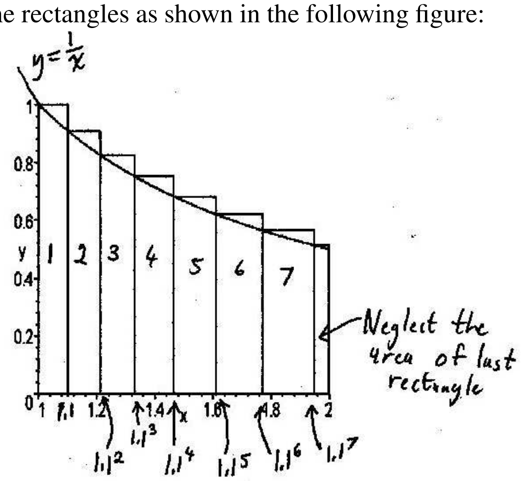
9.(1 pt) Compute an approximation to

$$\int_1^2 \frac{1}{x}\, dx,$$

which gives the area under $y = \frac{1}{x}$ for $1 \le x \le 2$, using a modified Riemann sum with the (NOT equally spaced) partition

$$1, 1.1, 1.1^2, 1.1^3, 1.1^4, 1.1^5, 1.1^6, 1.1^7, 2$$

and left hand endpoints EXCEPT neglecting the area of the last rectangle. This amounts to computing the sum of the areas of the rectangles as shown in the following figure:



As you can see in the figure, the area of the last rectangle is relatively small compared to the others, and the other rectangles already give an overestimate of the area.

Please note that the problem is NOT asking for the value of $\int_1^2 \frac{1}{x}\, dx$. Rather it is asking for the EXACT values of the areas of the 7 approximating rectangles and for the EXACT value of the sum of the areas of the rectangles. Calculator approximations (no matter how accurate) will NOT be accepted. Do the calculations by hand using fractions (until you notice the pattern in the areas).

The area of the first rectangle = ______

The area of the second rectangle = ______

The area of the third rectangle = ______

The area of the fourth rectangle = ______

The area of the fifth rectangle = ______

The area of the sixth rectangle = ______

The area of the seventh rectangle = ______

The sum of the areas of the 7 rectangles = ______

10.(1 pt) This problem is a reprise of problem 5 with 1.1 replaced by 1.001

Compute an approximation to

$$\int_1^3 \frac{1}{x}\, dx,$$

which gives the area under $y = \frac{1}{x}$ for $1 \le x \le 3$, using a modified Riemann sum with the (NOT equally spaced) partition

$$1, 1.001, 1.001^2, 1.001^3, \ldots, 1.001^N, 3$$

and left hand endpoints EXCEPT neglecting the area of the last rectangle. Here N denotes the largest possible power which fits in the interval $1 \le x \le 3$.

Please note that the problem is NOT asking for the value of $\int_1^3 \frac{1}{x}\, dx$. Rather it is asking for the EXACT values of the areas of approximating rectangles and for the EXACT value of the sum of the areas of the rectangles. Calculator approximations (no matter how accurate) will NOT be accepted. Do the calculations by hand using fractions (until you notice the pattern in the areas).

The number of approximating rectangles is:

$N =$ ______

The area of the first rectangle = ______

The area of the second rectangle = ______

The area of the third rectangle = ______

The sum of the areas of the N rectangles = ______

11.(1 pt) This problem is a reprise of problem 6 with 1.001 replaced by 1.000001

Compute an approximation to

$$\int_1^6 \frac{1}{x}\, dx,$$

which gives the area under $y = \frac{1}{x}$ for $1 \le x \le 6$, using a modified Riemann sum with the (NOT equally spaced) partition

$$1, 1.000001, 1.000001^2, 1.000001^3, \ldots, 1.000001^N, 6$$

and left hand endpoints EXCEPT neglecting the area of the last rectangle. Here N denotes the largest possible power which fits in the interval $1 \le x \le 6$.

Please note that the problem is NOT asking for the value of $\int_1^6 \frac{1}{x}\, dx$. Rather it is asking for the EXACT values of the areas of approximating rectangles and for the EXACT value of the sum of the areas of the rectangles. Calculator approximations (no matter how accurate) will NOT be accepted. Do the calculations by hand using fractions (until you notice the pattern in the areas).

The number of approximating rectangles is:

$N =$ ______

The area of the first rectangle = ______

The area of the second rectangle = ______

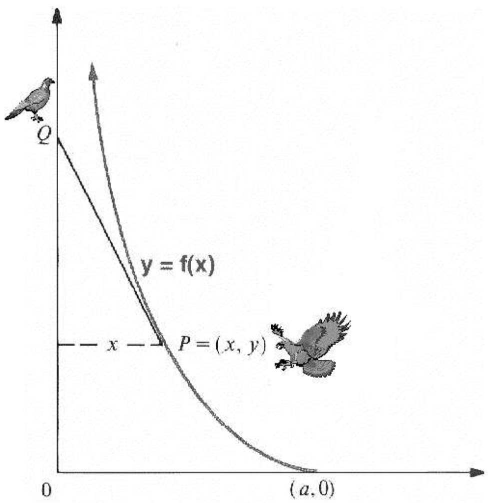
The area of the third rectangle = ______

The sum of the areas of the N rectangles = ______

The ingenious idea of using these unusual nonequally spaced partitions to compute the area under $y = \frac{1}{x}$ thus relating it to Napier's logarithms is due to a Belgian monk, Gregory St. Vincent around 1647.

12.(1 pt) Suppose that a hawk, whose initial position is $(a,0) = (8000,0)$ on the x-axis, spots a pigeon at $(0,1000)$ on the y-axis. Suppose that the pigeon flies at a constant speed of 10 ft/sec in the direction of the y-axis (oblivious to the hawk), while the hawk flies at a constant speed of 40 ft/sec, always in the direction of the pigeon.

The problem is to find an equation for the flight path of the hawk (the curve of pursuit) and to find the time and place where the hawk will catch the pigeon. Assume that in this problem all distances are measured in feet and all times measured in seconds. Leave out all dimensions from your answers.



Consider the diagram above (click on it for a better view) which represents the situation at an arbitrary time t during the pursuit. The points P and Q represent the positions of the hawk and pigeon respectively at that time instant t, with $y = f(x)$ representing the flight path of the hawk.

The pigeon's position $Q = (0, g(t))$ is given by the following function of time

$g(t)$ = ______

The fact that the hawk is always headed in the direction of the pigeon means that the line PQ is tangent to the pursuit curve $y = f(x)$. This tells us that $\frac{dy}{dx} = h(x,y,t)$ where

$h(x,y,t)$ = ______

(Your answer must involve the three variables x, y, and t.)

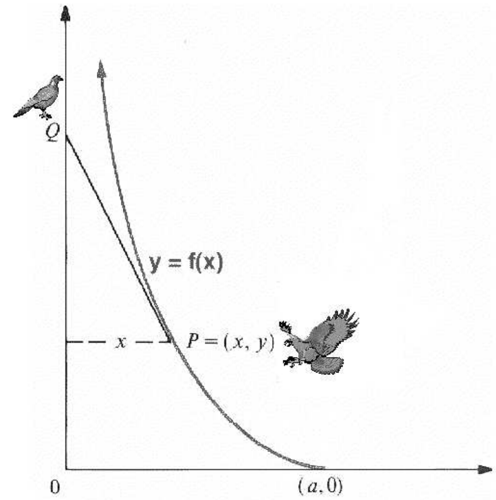
If we solve the equation

$$p = h(x,y,t),$$

where $p = \frac{dy}{dx}$, for time we obtain that $t = k(x,y,p)$ where

$k(x,y,p) =$ = ______

(Your answer must involve the three variables x, y, and p, where p stands for $\frac{dy}{dx}$.)



13.(1 pt)

Again referring to the diagram above (click on it for a better view) we see that the distance that the hawk has flown in time t is given by the integral $\int_c^d F dx$ where

c = ______

d = ______

(Hints: Note that this integral computes the length of a curve. Also recall that the hawk's initial position is at $(a,0) = (8000,0)$.)

and

F = ______

(Use p to denote $\frac{dy}{dx}$ in your last answer above.)

On the other hand the hawk is flying at a constant speed of 40 for time t. Hence the total distance it has flown is ______ .

If we equate this to the distance we just computed and solve for t we obtain

$$t = \int_c^d G dx$$

where G = ______

(Remember to use p to represent $\frac{dy}{dx}$.)

14.(1 pt) Equating the two expressions for t from Problems 1 and 2 we obtain the integral equation

$$k(x,y,p) = \int_c^d G dx$$

To get rid of the integral, we differentiate both sides of the equation with respect to x. On the left hand side of the resulting equation we obtain the following expression (which might involve x, y, $p = \frac{dy}{dx}$ and $q = \frac{dp}{dx} = \frac{d^2y}{dx^2}$

______

(remember to separate different variables in a product with spaces or multiplication signs)

while on the right hand side (after applying the Fundamental Theorem of Calculus) we obtain

______

The resulting differential equation we obtained above is a separable differential equation in the variables p and x. We can rewrite it in the form

$$K(p)dp = L(x)dx$$

(with all numerical factors moved to the right hand side of the equation so that L(1)=10/40.) where

$K(p)$ = ______

$L(x)$ = ______

15.(1 pt) Integrating the left hand side of the equation $\int K(p)dp$, using the methods of sections 7.3 and 7.3, we obtain

$\int K(p)dp$ = ______

while on the right hand side we obtain $\int L(x)dx$ = ______ $+C$

Plugging in the initial positions of the hawk and pigeon, and recalling that $p = \frac{dy}{dx}$ is the slope of the tangent line, we find that

C = ______

16.(1 pt) Solving the equation we obtained in Problem 4 for p in terms of x, we obtain

p = ______

(Hints for solving for p: Exponentiate to get rid of the logarithm. Then isolate the square root on one side of the equation and square both sides.)

Recalling that $p = \frac{dy}{dx}$ and integrating. we obtain that

y = ______ $+C$

Plugging in the initial position of the hawk we obtain that the constant of integration is given by

C = ______

17.(1 pt) Hence the hawk catches the pigeon at the point $(0, c)$ where

c = ______

at time t = ______