Math 217 - Practice Final

Notes to the reader: This practice exam is meant to help you prepare for the final exam by showing you a variety of problems that will cover many of the concepts to learn for the exam. However, that does not mean that these are the only problems to study for the test - in general, these problems may or may not be similar to what appears on the exam. You should also not use these problems or this practice as a gauge of the actual exam’s difficulty. I strongly encourage you to try these problems without using a calculator, your notes, or any other references.

The actual exam will consist of 18 multiple choice questions worth 5 – 7% each. You should make sure that you can write clear and concise solutions to each of these problems!

This practice document only contains problems from the relevant sections of Chapter 4, 5, and 6 which have been covered since Exam 3. Although this document is not cumulative, the final exam is cumulative. Any material covered in the semester is fair for the exam, and there will be at least one problem from every chapter we covered.

For the exam, you are allowed a double sided 4 x 6 inch notecard on which you may hand-write (not type) any information you would like. You must put your name on the card and turn it in with your exam. I will give you the same Laplace transform table as Exam 3.

1. Solve the system \( x' = 4x + y + 2t, \ y' = -2x + y \) with initial conditions \( x(0) = 1, y(0) = 3 \).

2. Find the general solution of the system

\[
\vec{x}' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \vec{x}.
\]

Then find the solution satisfying \( x_1(0) = 10, x_2(0) = 12, x_3(0) = -1 \).

3. Use eigenvalues and eigenvectors to find two complex solutions of

\[
\vec{x}' = \begin{bmatrix} 7 & -5 \\ 4 & 3 \end{bmatrix} \vec{x}.
\]

Then find two linearly independent real-valued solutions.

4. Find the general solution of the system

\[
\vec{x}' = \begin{bmatrix} 5 & 1 \\ -1 & 3 \end{bmatrix} \vec{x}.
\]

5. Draw a careful phase portrait and identify the shape (as conic sections) of the trajectories of \( x' = 3y, y' = x \).

6. Analyze the stability of the unique critical point of \( x' = x - 5y - 5, y' = x - y - 3 \).

7. Analyze the stability of the critical point(s) of \( x' = 30x - 3x^2 + xy, y' = 60y - 3y^2 + 4xy \), and use the analysis to draw a phase portrait. If these are two species populations, describe how they evolve over time (do they go extinct, grow, cooperate, etc.?).
Answers

Comment: WolframAlpha and Geoebra are great tools for visualizing phase portraits, as are Matlab and Mathematica, so I strongly encourage you to develop your own plots with them.

1. \( x(t) = \frac{1}{18}(-6t - 81e^{2t} + 98e^{3t} + 1), y(t) = \frac{1}{9}(-6t + 81e^{2t} - 49e^{3t} - 5). \)

2. The particular solution is \([7e^{2t} + 3e^{-t}, 7e^{2t} + 5e^{-t}, 7e^{2t} - 8e^{-t}]^T\).

3. The complex solutions are

\[
\begin{bmatrix}
1 + 2i \\
2
\end{bmatrix} e^{(5+4i)t}
\]

and its conjugate. Two real-valued solutions are

\[
e^{5t}\begin{bmatrix}
\cos 4t - 2\sin 4t \\
2\cos 4t
\end{bmatrix}
\text{ and }
\begin{bmatrix}
2\cos 4t + \sin 4t \\
\sin 4t
\end{bmatrix}.
\]

4. One way to write the general solution is \(c_1\hat{v}e^{4t} + c_2(\hat{v} + \hat{w})e^{4t}\) with \(\hat{v} = [1, -1]^T\) and \(\hat{w} = [1, 0]^T\).

5. The trajectories are hyperbolic, because they are the level curves of \(z = x^2 - 3y^2\). To see this, notice that

\[
\frac{d}{dt}(x^2 - 3y^2) = 2(xx' - 3yy') = 2(x \cdot 3y - 3y \cdot x) = 0.
\]

6. It’s a stable center.

7. See Section 6.3 #32. This describes logistic populations (e.g. \(x' = x(30 - 3x) + xy\)) with cooperation. The origin is a source, there are two saddle points (corresponding to one population being extinct and the other stable), and a sink with non-zero coexisting populations.