Solutions

Math 217 - Exam 2

9 October 2018

Instructions: For problems 1-9, select the correct answer from the choices provided. You do not need to show any work for these problems, and the grading is only based on the option selected.

For problems 10-12, provide a *complete* solution, showing how you arrive at the final answer. An answer without work will only receive partial credit.

The exam is worth 54+46 = 100 points. No calculators, electronic devices, or notesheets are allowed on this exam.

Multiple Choice Section

1. (6 points) What is the Wronskian of $f_1(x) = x$, $f_2(x) = x^2$, and $f_3(x) = x^3$?

   (a) 0
   (b) $2x^3$
   (c) $6x^3$
   (d) $10x^3$
   (e) None of the above

$$\det \begin{bmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{bmatrix} = x \det \begin{bmatrix} 2x & 3x^2 \\ 2 & 6x \end{bmatrix} - 1 \det \begin{bmatrix} x^2 & x^3 \\ 2 & 6x \end{bmatrix} + 0$$

$$= x\,(2x \cdot 6x - 3x^2 \cdot 2) - (x^2 \cdot 6x - x^3 \cdot 2)$$

$$= x\,(12x^2 - 6x^2) - (6x^3 - 2x^3)$$

$$= 6x^3 - 4x^3 = \underline{\underline{2x^3}}.$$

2. (6 points) Which of the following equations has

$$y = \underbrace{C_1 + C_2x + C_3x^2 + C_4x^3}_{\substack{r=0 \\ \text{mult } 4}} + \underbrace{C_5e^{2x}\cos x + C_6e^{2x}\sin x}_{r = 2 \pm i}$$

as its general solution?

(a) $y^{(6)} = 0$

(b) $y' + y'' + y^{(3)} + y^{(4)} = 0$

(c) $y' + y'' + y^{(3)} + y^{(4)} + 4y^{(5)} - 5y^{(6)} = 0$

(d) $y^{(6)} - 2y^{(5)} + 5y^{(4)} = 0$

(e) $y^{(6)} - 4y^{(5)} + 5y^{(4)} = 0$

$$r^4 \left(r - (2+i)\right)\left(r - (2-i)\right)$$
$$= r^4 \left((r-2) - i\right)\left((r-2) + i\right)$$
$$= r^4 \left[(r-2)^2 - i^2\right]$$
$$= r^4 \left(r^2 - 4r + 5\right)$$
$$= r^6 - 4r^5 + 5r^4$$

3. (6 points) Which of the following sets of functions are linearly dependent?

(a) $\sin^2 x, \cos^2 x, 1$

(b) $0, 1, x$

(c) $x^2, x^2 - 1, x^2 + 1$

(d) (a) and (b) only

(e) (a) and (b) and (c)

$1 \sin^2 x + 1 \cos^2 x - 1 \cdot 1 = 0$

$1 \cdot 0 + 0 \cdot 1 + 0 \cdot x = 0$

$-2 \cdot x^2 + 1(x^2 - 1) + 1(x^2 + 1) = 0$

All.

4. (6 points) Consider the equation

$$(\sin t)y'' + e^t y = e^t \ln(t)$$

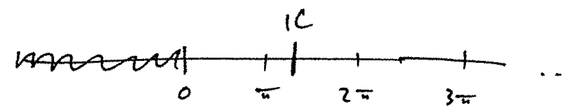
with initial conditions $y(4) = 0, y'(4) = 0, y''(4) = 0$. On what interval is a unique solution guaranteed to exist?

(a) $(\pi, 2\pi)$
(b) $(0, 2\pi)$
(c) $(0, \infty)$
(d) $(-\infty, \infty)$
(e) No solution exists

$$y'' = -\frac{e^t}{\sin t} y + \frac{e^t \ln t}{\sin t}$$

Bad at $n\pi$, $n \in \mathbb{Z}$

Bad at $n\pi$, $n \in \mathbb{Z}$ and $t \le 0$.



5. (6 points) Consider the equation $y'' + 4y' + 13y = 0$ with initial conditions $y(0) = 0, y'(0) = 4$. What is $y(\pi)$?

(a) -1
(b) 0
(c) 1
(d) e
(e) e^{-3}

$r^2 + 4r + 13 = 0$
$(r+2)^2 + 3^2 = 0$
$r = -2 \pm 3i$

$y = Ae^{-2x}\cos 3x + Be^{-2x}\sin 3x$

$0 = y(0) = 1A + 0B \longrightarrow A = 0.$

Notice $\sin 3\pi = 0.$

6. (6 points) Given the equation

$$y'' + 6y' + 34y = e^{-3x}\cos(5x),$$

what would the candidate particular solution be in the method of undetermined coefficients?

(a) 0

(b) $Ae^{-3x}\cos(5x)$

(c) $Ae^{-3x} + B\cos(5x) + C\sin(5x)$

(d) $Ae^{-3x}\cos(5x) + Be^{-3x}\sin(5x)$

(e) $Axe^{-3x}\cos(5x) + Bxe^{-3x}\sin(5x)$

$r^2 + 6r + 34 = 0$

$(r+3)^2 + 5^2 = 0$

$r = -3 \pm 5i$

Duplication $\longrightarrow$ multiply by x.

7. (6 points) Which of the following is the particular solution to the equation $y'' + y = x^2$?

(a) 2

(b) $x^2 - 2$

(c) x^2

(d) $x^2 + x - 2$

(e) $x^2 + 2x - 2$

$y_p = Ax^2 + Bx + C$

$y_p'' = 2A$

$\therefore \quad 2A + (Ax^2 + Bx + C) = x^2$

$Ax^2 + Bx + (2A + C) = 1x^2 + 0x + 0$

$A = 1 \qquad B = 0 \qquad 2A + C = 0$

$\hookrightarrow C = -2$

10. (**Full Answer Question**) A certain spring stretches 1 m under a force of 4 N. We attach the spring to a block with mass 1 kg. The block is moved 1 m left and is released with a velocity of 5 m/s to the right.

(a) (7 points) Find the position $x(t)$ of the block at time t.

(b) (4 points) Write the solution in the form $A\cos(\omega_0 t - \delta)$. You may find it helpful to recall that $\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$.

a) $k = 4N/1m \longrightarrow mx'' + kx = 0 \longrightarrow x'' + 4x = 0$

$x = C_1 \cos 2t + C_2 \sin 2t$

$x(0) = -1 \longrightarrow C_1 = -1$

$x'(0) = 5 \longrightarrow C_2 = 5/2$

$x(t) = -\cos 2t + \frac{5}{2}\sin 2t$

b) $x(t) = -1\cos 2t + \frac{5}{2}\sin 2t$

$x(t) = A\cos\delta \cos 2t + A\sin\delta \sin 2t$

$A\cos\delta = -1$

$A \sin\delta = 5/2$

Sin > 0, cos < 0 $\longrightarrow$ 2nd Quad.

Square & add

$A^2 = 1 + 25/4$

$A = \sqrt{29/4}$.

Divide. $\frac{\sin\delta}{\cos\delta} = \frac{5/2}{-1}$

$\delta = \arctan(-5/2) + \pi$

$$\boxed{x(t) = \sqrt{\frac{29}{4}}\cos\left(2t - (\arctan(-5/2) + \pi)\right)}$$

8. (6 points) What is the general solution to $y''' - 4y'' - y' + 4y = 0$?

(a) $C_1e^{-x} + C_2xe^{-x} + C_3x^2e^{-x}$
(b) $C_1 + C_2x + C_3x^2$
(c) $C_1e^{-x} + C_2e^{x} + C_3e^{4x}$
(d) $C_1e^{-4x} + C_2e^{-x} + C_3e^{4x}$
(e) $C_1 + C_2e^{-4x} + C_3e^{-x} + C_4e^{4x}$

$r^3 - 4r^2 - r + 4 = 0$

$r^2(r-4) - (r-4) = 0$

$(r^2 - 1)(r - 4) = 0$

$r = -1, 1, 4.$

9. (6 points) What is the general solution to $y''' - 6y'' + 12y' - 8y = 0$?

(a) $C_1e^{2x} + C_2xe^{2x} + C_3x^2e^{2x}$
(b) $C_1e^{-2x} + C_2xe^{-2x} + C_3x^2e^{-2x}$
(c) $C_1e^{2x} + C_2e^{2x} + C_3e^{2x}$
(d) $C_1e^{-6x} + C_2e^{12x} + C_3e^{-8x}$
(e) None of the above

$r^3 - 6r^2 + 12r - 8 = 0$

Possible rational roots: $\pm 1, \pm 2, \pm 4, \pm 8$.

2 is a root.

$$\begin{array}{r|l} & r^2 - 4r + 4 \\ \hline (r-2) & r^3 - 6r^2 + 12r - 8 \\ & r^3 - 2r^2 \\ \hline & -4r^2 + 12r \\ & -4r^2 + 8r \\ \hline & 4r - 8 \\ & 4r - 8 \\ \hline & 0 \end{array}$$

Note $r^2 - 4r + 4 = (r-2)^2$

$\therefore r = 2$, mult. 3.

11. (15 points) (**Full Answer Question**) Solve the initial value problem

$$y'' - 4y = 4e^{2x} + 4$$

with initial conditions $y(0) = 0$, $y'(0) = 5$.

$y_c'' - 4y_c = 0 \qquad r^2 - 4 = 0 \qquad r = \pm 2$

$y_c = C_1 e^{2x} + C_2 e^{-2x}$

$y_p = Axe^{2x} + B$ because e^{2x} solves homogeneous eqn.

$y_p' = Ae^{2x} + 2Axe^{2x}$

$y_p'' = 2Ae^{2x} + 2Ae^{2x} + 4Axe^{2x}$

$(\cancel{4Axe^{2x}} + 4Ae^{2x}) - 4(\cancel{Axe^{2x}} + B) = 4e^{2x} + 4$

$\therefore \quad 4A = 4, \quad -4B = 4 \longrightarrow A = 1, \; B = -1$

$y_p = xe^{2x} - 1$

General solution $\quad y = C_1 e^{2x} + C_2 e^{-2x} + xe^{2x} - 1$

$0 = y(0) = C_1 + C_2 + 0 - 1$

$5 = y'(0) = 2C_1 - 2C_2 + 1$

$\therefore \quad C_1 + C_2 = 1$

$2C_1 - 2C_2 = 4$

$C_1 + C_2 = 1$

$C_1 - C_2 = 2$

add $\quad 2C_1 = 3 \longrightarrow C_1 = 3/2,$

$C_2 = -1/2$

$$\boxed{y = \frac{3}{2}e^{2x} - \frac{1}{2}e^{-2x} + xe^{2x} - 1}$$

12. (**Full Answer Question**) A certain spring-mass-dashpot system is modeled by

$$x'' + 2px' + \omega_0^2 x = 5\cos t$$

with damping constant $p = 1$ and circular frequency $\omega_0 = 1$. The mass starts at equilibrium ($x(0) = 0$) moving to the right ($x'(0) = 5$).

(a) (10 points) Find the solution $x(t)$.

(b) (5 points) Identify the transient solution and the long-term behavior. What value is the amplitude tending towards?

(c) (5 points) Suppose we remove the dashpot and the damping constant is now $p = 0$. What happens to the limiting amplitude?

a) $x'' + 2x' + x = 5\cos t$

$r^2 + 2r + 1 = 0 \longrightarrow (r+1)^2 = 0 \qquad r = -1,$ mult. 2

$x_c = C_1 e^{-t} + C_2 t e^{-t}$

Particular: $x_p = A\cos t + B\sin t$

$x_p' = -A\sin t + B\cos t$

$x_p'' = -A\cos t - B\sin t$

$(-\cancel{A\cos t} - \cancel{B\sin t}) + 2(-A\sin t + B\cos t) + \cancel{(A\cos t + B\sin t)} = 5\cos t$

$\therefore \quad -2A = 0, \quad 2B = 5$

$\therefore \quad x_p = \frac{5}{2}\sin t$

General: $x = \cancel{C_1 e^{-t}}^{\,0} + C_2 t e^{-t} + \frac{5}{2}\sin t$

$0 = x(0) = C_1$

$5 = x'(0) = C_2(-te^{-t} + e^{-t})\big|_{t=0} + \frac{5}{2}$

$5 = C_2 + 5/2 \qquad C_2 = 5/2.$

$\therefore \quad \boxed{x(t) = \frac{5}{2}te^{-t} + \frac{5}{2}\sin t}$

Continued

Extra work area

b) Transient: $\frac{5}{2} t e^{-t}$

Long term / steady periodic: $\frac{5}{2} \sin t$.

Limiting amplitude: $5/2$.

c) We'd have $x'' + x = 5 \cos t$. The resonant frequency is 1, as is the forcing frequency. The particular solution is of the form $A t \sin t + B t \cos t$ so the amplitude blows up.

Extra work area