Solutions

Math 217 - Exam 3

13 November 2018

Instructions: For problems 1-10, select the correct answer from the choices provided. You do not need to show any work for these problems, and the grading is only based on the option selected.

For problems 11-13, provide a *complete* solution, showing how you arrive at the final answer. An answer without work will only receive partial credit.

The exam is worth 60+40 = 100 points. No calculators, electronic devices, or notesheets are allowed on this exam.

You may use without proof the following Laplace transform identities:

(1) $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$ provided that the integral converges

(2) $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$ and $\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$

(3) $\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s)$

(4) $\mathcal{L}\{(f * g)(t)\} = F(s)G(s)$ holds for the convolution of two functions

(5) $\mathcal{L}\{tf(t)\} = -F'(s)$

(6) $\mathcal{L}\{t^n\} = n!/s^{n+1}$

(7) $\mathcal{L}\{e^{at}\} = 1/(s-a)$ and $\mathcal{L}\{t^n e^{at}\} = n!/(s-a)^{n+1}$

(8) $\mathcal{L}\{\sin(kt)\} = k/(s^2+k^2)$ and $\mathcal{L}\{\cos(kt)\} = s/(s^2+k^2)$

(9) $\mathcal{L}\{u(t-a)\} = e^{-as}/s$ and $\mathcal{L}\{\delta(t-a)\} = e^{-as}$

1. (6 points) Choose the Laplace transform of the function

$$f(t) = \begin{cases} t & 0 \le t < 3 \\ 0 & 3 \le t \end{cases}$$

D

(a) $1/s^2$
(b) e^{-3s}/s^2
(c) $(1 + e^{-3s})/s^2$
(d) $(1 - e^{-3s} - 3se^{-3s})/s^2$
(e) None of the above

$$f(t) = t\,[1 - u(t-3)]$$
$$= t - (t-3)\,u(t-3) - 3u(t-3).$$
$$\downarrow \qquad \downarrow \qquad \downarrow$$
$$1/s^2 \qquad e^{-3s}/s^2 \qquad 3e^{-3s}/s.$$

2. (6 points) Choose the Laplace transform of $t \sin t$.

B

(a) $1/(s^2(s^2+1))$
(b) $2s/(s^2+1)^2$
(c) $-2s/(s^2+1)^2$
(d) The transform does not exist
(e) None of the above

$$\mathcal{L}\{t \sin t\} = -\frac{d}{ds}\mathcal{L}\{\sin t\}$$
$$= -\frac{d}{ds}\frac{1}{s^2+1}$$
$$= -\frac{-2s}{(s^2+1)^2}$$

3. (6 points) Choose the inverse Laplace transform of

$$F(s) = \frac{2s+1}{s(s^2+9)}. \quad = \frac{A}{s} + \frac{Bs+C}{s^2+9}$$

E

(a) 0
(b) $\frac{1}{9} - \frac{1}{9}\cos(3t)$
(c) $\frac{1}{9} - \frac{1}{9}\cos(3t) + 2\sin(3t)$
(d) $1 + \cos(3t) - 2\sin(3t)$
(e) None of the above

$$2s+1 = As^2 + 9A + Bs^2 + Cs$$

$$\therefore \quad C = 2, \quad A = 1/9, \quad B = -A = -1/9.$$

$$F(s) = \frac{1}{9s} - \frac{1}{9}\frac{s}{s^2+9} + \frac{2}{3}\frac{3}{s^2+9}$$

$$f(t) = \frac{1}{9} - \frac{1}{9}\cos 3t + \frac{2}{3}\sin 3t$$

4. (6 points) If w is a function such that

$$\int_0^t w(\tau)e^{2(t-\tau)}\,d\tau = \sin(t)$$

for all t, find w.

C

(a) $e^{-2t}\sin(t)$
(b) $\cos(t)$
(c) $\cos(t) - 2\sin(t)$
(d) $\cos(t) + 2\sin(t)$
(e) No such w exists

$$w * e^{2t} = \sin t$$

$$W(s)\frac{1}{s-2} = \frac{1}{s^2+1}$$

$$W(s) = \frac{s}{s^2+1} - 2\frac{1}{s^2+1}$$

5. (6 points) If x is a solution to $tx'' + x = 0$ satisfying $x(0) = 0$, which is an equation for the Laplace transform?

C

(a) $\frac{1}{s^2}X(s) + X(s) = 0$
(b) $s^2X'(s) + X(s) = 0$
(c) $s^2X'(s) + (2s-1)X(s) = 0$
(d) Not enough information given
(e) None of the above

$$\mathcal{L}\{tx''\} = -\frac{d}{ds}\mathcal{L}\{x''\}$$
$$= -\frac{d}{ds}\left(s^2X(s) - \cancel{sx(0)}^{0} - x'(0)\right)$$
$$= -2sX(s) - s^2X'(s)$$
$$\therefore\ -2sX(s) - s^2X'(s) + X(s) = 0.$$

6. (6 points) Find the solution to $x'' + 5x' + 4x = \delta(t-2)$ subject to $x(0) = x'(0) = 0$.

E

(a) $x(t) = 0$ for all t
(b) $\frac{1}{3}e^{-4t} - \frac{1}{3}e^{-t}$
(c) $\frac{1}{3}\left(e^{-4t} - e^{-t}\right)u(t-2)$
(d) $\frac{1}{3}\left(e^{8}\cdot e^{-4t} - e\cdot e^{-t}\right)u(t-2)$
(e) None of the above

$$(s^2+5s+4)X(s) = e^{-2s}$$
$$X(s) = \frac{e^{-2s}}{(s+1)(s+4)}$$

$$\frac{1}{(s+1)(s+4)} = \frac{A}{s+1} + \frac{B}{s+4}$$

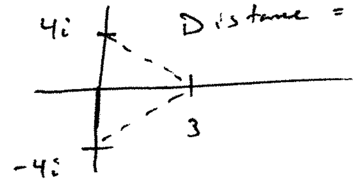
$$A(s+4) + B(s+1) = 1$$
$$s = -1 \longrightarrow A = 1/3$$
$$s = -4 \longrightarrow B = -1/3$$

$$\longrightarrow X(s) = \frac{1}{3}\left[\frac{1}{s+1} - \frac{1}{s+4}\right]e^{-2s}$$
$$x(t) = \frac{1}{3}\left[e^{-(t-2)} - e^{-4(t-2)}\right] \cdot u(t-2).$$
$$= \frac{1}{3}\left[-e^{8}\cdot e^{-4t} + e^{2}\cdot e^{-t}\right]u(t-2)$$

7. (6 points) Given the equation $(x^2+16)y''+e^x y=0$ with initial conditions $y(3)=3, y'(3)=6$, consider the power series solution centered at $x=3$. What is the largest interval on which the series is guaranteed to converge?

D

(a) Only converges at $x=3$
(b) $(-4,4)$
(c) $(-1,7)$
(d) $(-2,8)$
(e) $\mathbb{R}$



Distance $= \sqrt{3^2+4^2} = 5$

$\therefore \ (3-5, \ 3+5).$

8. (6 points) Given $(x^2-1)y''+4xy'+2y=0$, find a formula relating the coefficients of $y=\sum_{n=0}^{\infty} c_n x^n$.

A

(a) $c_{n+2}=c_n$
(b) $(n+2)(n+1)c_{n+2}-(n+2)(n+1)c_n+4(n+1)c_{n+1}+2c_n=0$
(c) $(n+1)(n+2)c_{n+2}=n(n-1)c_n+2c_n-4(n+1)c_{n+1}$
(d) Not enough information given
(e) No power series solution exists

$$x^2y'' = \sum_{n=0}^{\infty} n(n-1)c_n x^n$$

$$-y'' = -\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n$$

$$4xy' = 4\sum_{n=0}^{\infty} n c_n x^n$$

$$2y = 2\sum_{n=0}^{\infty} c_n x^n$$

Add all 4, get

$$n(n-1)c_n - (n+2)(n+1)c_{n+2} + 4nc_n + 2c_n = 0$$

$$\therefore \ c_{n+2} = \frac{n(n-1)+4n+2}{(n+2)(n+1)} c_n = \frac{n^2+3n+2}{n^2+3n+2} c_n = c_n$$

9. (6 points) For the equation $x^2(1-x^2)y''+2xy'-2y=0$, determine if $x=0$ is a regular singular point. If it is, what are the exponents of the differential equation at $x=0$?

(a) 0 is not a regular singular point

(b) $r=1,-2$

(c) $r=2,-1$

(d) $r=-1,-1$

(e) None of the above

B

$$y'' + \frac{2/1-x^2}{x} y' + \frac{-2/1-x^2}{x^2} y = 0$$

$$p = \frac{2}{1-x^2} \longrightarrow p_0 = 2 \qquad \text{Regular!}$$

$$q = -\frac{2}{1-x^2} \longrightarrow q_0 = -2$$

Indicial polynomial: $r(r-1) + 2r - 2 = 0$

$$r^2 + r - 2 = 0$$

$$(r+2)(r-1) = 0$$

10. (6 points) Consider the Airy equation $y''+xy=0$ with initial conditions $y(0)=1, y'(0)=0$. Find the first four non-zero terms of the power series solution.

(a) $1+\frac{2}{4!}x^3+\frac{2\cdot5}{7!}x^6+\frac{2\cdot5\cdot8}{10!}x^9$

(b) $1+\frac{1}{3!}x^3+\frac{1\cdot4}{6!}x^6+\frac{1\cdot4\cdot7}{9!}x^9$

(c) $1+\frac{2}{2!}x^2+\frac{2^2}{4!}x^4+\frac{2^3}{6!}x^6$

(d) Not enough information is given

(e) No power series solution exists

$\boxed{\text{Typo}}$

Should have had $y'' - xy$ instead.

11. (15 points) (**Full Answer Question**) A mass-spring system is given by $x'' + 4x = f(t)$ where the driving force is

$$f(t) = \begin{cases} 0 & 0 \le t \le 1 \\ e^{1-t} & 1 \le t \end{cases}$$

The mass starts at rest and at the equilibrium position.

(a) (7 points) Find the Laplace transform $X(s)$ of the solution.

(b) (8 points) Find the position $x(t)$.

$$f(t) = e^{-(t-1)} u(t-1)$$

a) $$F(s) = \frac{e^{-s}}{s+1}$$

$$\therefore \boxed{X(s) = \frac{e^{-s}}{(s+1)(s^2+4)}}$$

b) $$\frac{1}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4}$$

$$1 = A(s^2+4) + (Bs+C)(s+1)$$

$$\left.\begin{aligned} A + B &= 0 \\ B + C &= 0 \end{aligned}\right] \longrightarrow A = C$$

$$\left. 4A + C = 1 \right] \longrightarrow A = C = 1/5, \quad B = -1/5.$$

$$\therefore X(s) = \left[\frac{1}{5} \frac{1}{s+1} - \frac{1}{5} \frac{s}{s^2+4} + \frac{1}{5} \cdot \frac{1}{2} \frac{2}{s^2+4} \right] e^{-s}$$

$$\therefore \boxed{x(t) = \left[\frac{1}{5} e^{-(t-1)} - \frac{1}{5} \cos 2(t-1) + \frac{1}{10} \sin 2(t-1) \right] u(t-1)}$$

12. (10 points) (**Full Answer Question**) For many applications we need to set up a damped mass-spring system that has a particular response to a given driving force (e.g we're trying to design a radio tuner that will only allow the system to have a large response to particular stations, or particular frequencies). Suppose that we wish for the system to respond to a driving force $f(t)$ by having position function

$$x(t) = \int_0^t \tau e^{-\tau} f(t-\tau)\, d\tau.$$

Find coefficients A, B, and C such that the equation

$$A(t)x''(t) + B(t)x'(t) + C(t)x(t) = f(t)$$

with initial conditions $x(0) = x'(0) = 0$ will exhibit this response. You may wish to cite Duhamel's principle.

$$x = \begin{pmatrix} \text{weight} \\ \text{function} \end{pmatrix} * \begin{pmatrix} \text{driving} \\ \text{force} \end{pmatrix} \longleftarrow \text{Duhamel}$$

$$\Longrightarrow \quad x = w(t) * f(t) \quad \text{where} \quad w(t) = te^{-t}$$

$$\Longrightarrow \quad X(s) = W(s)F(s) = \frac{1}{(s+1)^2} F(s)$$

$$\Longrightarrow \quad (s+1)^2 X(s) = F(s)$$

$$\Longrightarrow \quad s^2 X(s) + 2sX(s) + X(s) = F(s)$$

$$\Longrightarrow \quad \boxed{\begin{array}{c} x''(t) + 2x'(t) + x(t) = f(t) \\ A = 1, \quad B = 2, \quad C = 1 \end{array}}$$

13. (15 points) (**Full Answer Question**) Consider the equation

$$y'' - xy' = 0$$

with initial conditions $y(0) = 0, y'(0) = 1$.

(a) (3 points) Find the radius of convergence of the power series solution.

(b) (8 points) Find and solve the recurrence relation for coefficients of the power series solution.

(c) (4 points) Give the solution to the initial value problem.

Comment: An answer with enough terms written out to CLEARLY identify the pattern will get full credit even without the factorial notation.

a) No singularities at all ... $\boxed{R = +\infty}$

b) $y = \sum_{n=0}^{\infty} c_n x^n$

$y' = \sum_{n=0}^{\infty} n c_n x^{n-1} \longrightarrow xy' = \sum_{n=0} n c_n x^n$

$y'' = \sum_{n=0 \text{ or } n=2}^{\infty} n(n-1) c_n x^{n-2} \longrightarrow y'' = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n$

$\therefore \quad 0 = y'' - xy' = \sum_{n=0}^{\infty} \left[(n+2)(n+1) c_{n+2} - n c_n\right] x^n$

Recurrence relation: $c_{n+2} = \dfrac{n c_n}{(n+2)(n+1)}$.

Note $c_0 = 0$, $c_1 = 1$, so no even terms!

$c_1 = 1 \longrightarrow c_3 = \dfrac{1 c_1}{3 \cdot 2} = \dfrac{1}{3 \cdot 2}$

$\longrightarrow c_5 = \dfrac{3 c_3}{5 \cdot 4} = \dfrac{3 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2}$

$\longrightarrow c_7 = \dfrac{5 c_5}{7 \cdot 6} = \dfrac{5 \cdot 3 \cdot 1}{7 \cdot 6 \cdot \dots \cdot 2} = \dfrac{5!!}{7!}$

$\boxed{c_{2n} = 0 \quad \text{So} \quad c_{2n+1} = \dfrac{(2n-1)!!}{(2n+1)!}}$

$\therefore \quad y = \sum_{n=0}^{\infty} \dfrac{(2n-1)!!}{(2n+1)!} x^{2n+1} = x + \dfrac{1}{3!} x^3 + \dfrac{3}{5!} x^5 + \dfrac{5 \cdot 3}{7!} x^7 + \dots$

Extra work area