Solutions

Happy holidays, and enjoy the break!!

Math 217 - Final Exam

- Tyler

14 December 2018

Instructions: This exam contains 18 multiple choice problems, each worth 5 points or 7 points each. You do not need to show any work for these problems, and the grading is only based on the option selected.

No calculators or other electronic devices are allowed on this exam. You are allowed one 4×6 index card, double-sided as a notesheet; it must be written in your own handwriting, and cannot be typed. Turn it in with your exam and write your name on it.

Table of Laplace Transform identities:

(1) $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$ provided that the integral converges

(2) $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$ and $\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$

(3) $\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s)$

(4) $\mathcal{L}\{(f * g)(t)\} = F(s)G(s)$ holds for the convolution of two functions

(5) $\mathcal{L}\{tf(t)\} = -F'(s)$

(6) $\mathcal{L}\{t^n\} = n!/s^{n+1}$

(7) $\mathcal{L}\{e^{at}\} = 1/(s-a)$ and $\mathcal{L}\{t^n e^{at}\} = n!/(s-a)^{n+1}$

(8) $\mathcal{L}\{\sin(kt)\} = k/(s^2+k^2)$ and $\mathcal{L}\{\cos(kt)\} = s/(s^2+k^2)$

(9) $\mathcal{L}\{u(t-a)\} = e^{-as}/s$ and $\mathcal{L}\{\delta(t-a)\} = e^{-as}$

1. (5 points) Consider the equation

$$x^2y' + 3xy = 4$$

$y' + \frac{3}{x} y = \frac{4}{x^2}$

with $y(1) = 2$. Compute $y(1/2)$.

(a) 1
(b) 2
(c) 3
(d) 4
(e) 5

$\mu(x) = e^{\int \frac{3}{x} dx} = e^{3 \ln x} = x^3$

$\hookrightarrow x^3 y' + 3x^2 y = 4x$

$(x^3 y)' = 4x$

$x^3 y = 2x^2 + C$

$y = \frac{2}{x} + \frac{C}{x^3}$

If $x = 1$, $y = 2 = 2 + C \Rightarrow C = 0$

$\therefore y(1/2) = \frac{2}{1/2} = 4.$

2. (5 points) Consider a solution $y(t)$ to the equation

$$y' = 2\sqrt{y}$$

which passes through both $(0, 0)$ and $(1, 1)$. What is the value of $y(-1)$?

(a) 0
(b) 1
(c) 2
(d) 3
(e) The given conditions are not enough to determine $y(-1)$.

$y' = f(y) = 2\sqrt{y}$

$\partial_y f = y^{-1/2}$ is not continuous when $y = 0$.

No existence-uniqueness.

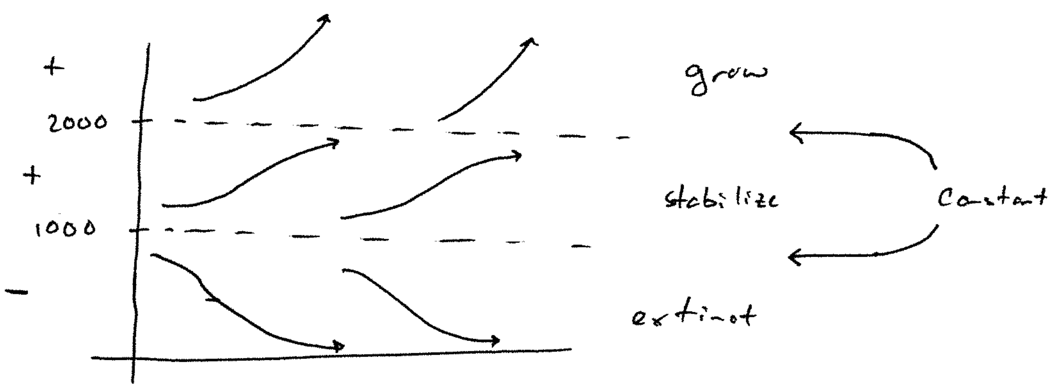
Explicit solutions: Constant 0, $(x-c)^2$.

3. (7 points) A certain population is modeled by the differential equation

$$\frac{dP}{dt} = (P - 1000)(P - 2000)^2.$$

Which of the following statements is true, given $P(0) > 0$?

(a) The population grows without bound.
(b) The population stabilizes at a (non-zero) equilibrium.
(c) The population oscillates between two equilibria.
(d) The population eventually goes extinct.
(e) Depending on the intial condition, the population can go extinct, grow without bound, or stay constant.



4. (5 points) Consider the solution of the equation

$$y'' - 2y' + y = 0 \longrightarrow r^2 - 2r + 1 = 0$$

subject to the initial conditions $y(0) = 0$, $y'(0) = 1$. Evaluate $y(1)$.

$(r-1)^2 = 0$

$r = 1$ mult 2

(a) $2e$
(b) e
(c) 1
(d) 0
(e) None of the above

$y = c_1 e^t + c_2 t e^t.$

$c_1 = y(0) = 0$

$c_2 = y'(0) = 1$

$\rightsquigarrow y(1) = e.$

5. (5 points) Consider the solution to

$$\frac{dy}{dx} = -\frac{2x+3y}{3x+2y}$$

which passes through the point $(1,1)$. If (x,y) is a point on the solution curve for which $x^2+y^2=2$, what is the value of xy?

(a) -1

(b) 0

(c) 1

(d) 2

(e) 3

$(3x + 2y)\,dx + (2x + 3y)\,dy = 0$

is exact with potential

$x^2 + y^2 + 3xy = C$; $x = 1, y = 1$

$\Rightarrow C = 5$

So if $x^2 + y^2 = 2$,

$2 + 3xy = 5$

$xy = 1$

6. (5 points) Consider the driven, damped, mass-spring system

$$mx'' + cx' + kx = \sin(\omega t)$$

with mass m, damping constant c, spring constant k, and an oscillatory driving force with frequency ω and amplitude 1. Which of the following statements are true?

(a) If $c = 0$, the amplitude is unbounded. - Not. if $\omega \neq \omega_0$

(b) If $c > 0$, the amplitude tends to zero. - Not with driving

(c) If $m = c = k = \omega = 1$, the system will exhibit resonance and the amplitude grows without bound. - Not resonant; natural frequency isn't 1.

(d) The system can only oscillate at frequency 1.

(e) None of the above are true.

Oscillates at frequency $\sqrt{\frac{k}{m}}$ too.

7. (5 points) Consider solving the equation

$$y'' + 4y = t\sin t + t$$

using the method of undetermined coefficients. How many unknown constants would the candidate particular solution require? Note: You do not need to find the particular solution.

(a) 3
(b) 4
(c) 5
(d) 6
(e) 7

$y_c = C_1 \cos 2t + C_2 \sin 2t \longrightarrow$ No duplication.

$y_p = A\, t\sin t + B\, t\cos t$
$+ C \sin t + D \cos t$
$+ Et + F.$

8. (5 points) If y_p is the particular solution to

$$y'' + y = \sin(t)$$

what is $y_p(2\pi)$?

(a) -2π
(b) $-\pi$
(c) 0
(d) π
(e) 2π

$y_p = A\, t\sin t + B\, t\cos t$

$y_p' = A(\sin t + t\cos t) + B(\cos t - t\sin t)$

$y_p'' = A(2\cos t - t\sin t) + B(-2\sin t - t\cos t)$

$\therefore\ y_p'' + y_p = 2A\cos t - 2B\sin t = \sin t$

$\therefore\ A = 0, \quad -2B = 1$

$\therefore\ y_p = -\frac{1}{2} t\cos t.$

$y_p(2\pi) = -\frac{1}{2}(2\pi) = -\pi.$

9. (5 points) If w is a function such that

$$\int_0^t e^{\tau} w(t-\tau)\, d\tau = te^t$$

for all $t \geq 0$, find $w(t)$.

(a) $w(t) = 1$
(b) $w(t) = t$
(c) $w(t) = e^t$
(d) $w(t) = t + e^t$
(e) $w(t) = te^t$

$$e^t * w = te^t$$

$$\frac{1}{s-1} W(s) = \frac{1}{(s-1)^2}$$

$$W(s) = \frac{1}{s-1}$$

$$w(t) = e^t$$

Check: $\int_0^t e^{\tau} e^{t-\tau}\, d\tau = e^t \int_0^t d\tau = te^t.$

10. (5 points) The solution $x(t)$ to the constant coefficient equation

$$x'' + ax' + bx = f(t); \quad x(0) = x'(0) = 0$$

is given by

$$x(t) = \int_0^t e^{-\tau} \sin \tau f(t-\tau)\, d\tau$$

for all $t \geq 0$. What is the value of $a \cdot b$?

(a) 0
(b) 1
(c) 2
(d) 3
(e) 4

$$X(s) = \mathcal{L}\{e^{-t} \sin t\} \cdot F(s)$$

$$= \frac{1}{(s+1)^2+1} F(s)$$

$$= \frac{1}{s^2+2s+2} F(s)$$

$$\therefore (s^2+2s+2) X(s) = F(s)$$

$$\therefore x'' + \underline{2}x' + \underline{2}x = f(t)$$

$$2 \cdot 2 = 4.$$

11. (5 points) The acceleration of a particle is given by

$$y'' = \begin{cases} 0 & \text{if } t < 1 \\ 3t & \text{if } t \geq 1 \end{cases}$$

with initial conditions $y(0) = y'(0) = 0$. Where is the particle at time 2?

(a) $y = 0.5$
(b) $y = 1$
(c) $y = 1.5$
(d) $y = 2$
(e) $y = 2.5$

$$y'' = 3t\, u(t-1) = 3(t-1)\,u(t-1) + 3u(t-1)$$

$$s^2 Y(s) = \frac{3e^{-s}}{s^2} + \frac{3e^{-s}}{s}$$

$$\therefore \quad Y(s) = \frac{3e^{-s}(1+s)}{s^4}.$$

$$\therefore \quad Y(s) = e^{-s}\left[\frac{3}{s^4} + \frac{3}{s^3}\right] = 3e^{-s}\left[\frac{1}{3!}\frac{3!}{s^4} + \frac{1}{2!}\frac{2!}{s^3}\right]$$

$$\therefore \quad y(t) = 3u(t-1)\left[\frac{1}{6}(t-1)^3 + \frac{1}{2}(t-1)^2\right].$$

$$y(2) = 3\left[\frac{1}{6} + \frac{1}{2}\right] = 3 \cdot \frac{4}{6} = 2.$$

12. (5 points) Consider the equation

$$(x^2 - 1)y'' - 4xy' + 6y = 0$$

with power series solution $y(x) = \sum_{n=0}^{\infty} c_n x^n$ and initial conditions $y(0) = 1, y'(0) = 0$. Evaluate the series $\sum_{n=0}^{\infty} c_n$.

$ = c_0 \qquad y'(0) = c_1$

Note: Explicit solution is $y = 1 + 3x^2$.

(a) 0
(b) 2
(c) 4
(d) 6
(e) 8

$$y = \text{—}$$

$$y' = \sum_{n=0}^{\infty} n c_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n.$$

$$\therefore \quad x^2 y'' - y'' - 4xy' + 6y$$

$$= \sum_{n=0}^{\infty} n(n-1) c_n x^n - \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n - 4\sum_{n=0}^{\infty} n c_n x^n + 6\sum_{n=0}^{\infty} c_n x^n$$

$$= \sum_{n=0}^{\infty} \left[n(n-1)c_n - (n+2)(n+1)c_{n+2} - 4n c_n + 6c_n\right] x^n = 0$$

$\therefore$ Recurrence is

$$(n^2 - n - 4n + 6)c_n = (n+2)(n+1)c_{n+2}$$

$$\therefore \quad c_{n+2} = \frac{n^2 - 5n + 6}{(n+2)(n+1)} c_n$$

$$\therefore \quad c_{n+2} = \frac{(n-2)(n-3)}{(n+2)(n+1)} c_n.$$

Odds are 0.

$c_0 = 1$, $c_2 = \left(\frac{6}{2} c_0\right) = 3$, $c_4 = 0$, $c_6 = 0$ etc.

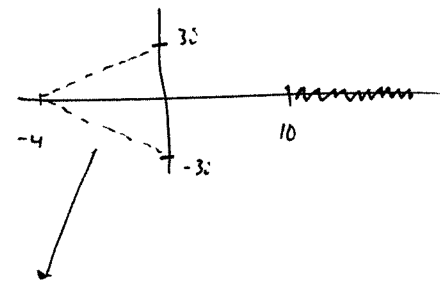
13. (5 points) Consider the differential equation

$$(x^2+9)y'' + (\ln(10-x))y' + y = 0$$

with initial conditions $y(-4) = 6, y'(-4) = 0$. Identify the interval of convergence of the power series solution centered at $x = -4$.

(a) $(-5, 5)$
(b) $(-7, -1)$
(c) $(-9, 1)$
(d) $(-18, 10)$
(e) $(-\infty, 10)$

Bad points : $x = \pm 3i, \quad x \geq 10.$



distance $= \sqrt{3^2 + 4^2} = 5$

$\therefore \quad (-4-5, \ -4+5)$

14. (5 points) Consider the equation $2x^2y'' + 5xy' + \cos(x)y = 0$. Which of the following statements about the series solution(s) is true?

(a) 0 is an ordinary point and we would use $\sum_{n=0}^{\infty} c_n x^n$.
(b) 0 is a regular singular point and we would use $\sum_{n=0}^{\infty} c_n x^n$.
(c) 0 is a regular singular point and we would use $\sum_{n=0}^{\infty} c_n x^{n+r}$ with $r = -1/2$ or $r = -1$.
(d) 0 is a regular singular point and we would use $\sum_{n=0}^{\infty} c_n x^{n+r}$ with $r = 1$ or $r = -1$.
(e) 0 is an irregular singular point and we would not use a series solution.

$$y'' + \frac{5/2}{x} y' + \frac{\cos x/2}{x^2} y = 0$$

$P = \frac{5}{2}$

$Q = \frac{1}{2}\cos x$ $\longmapsto$ Indicial Polynomial

$r(r-1) + \frac{5}{2}r + \frac{1}{2} = 0$

$r^2 + \frac{3}{2}r + \frac{1}{2} = 0$

$(r + \frac{1}{2})(r+1) = 0$

$r = -\frac{1}{2}, \quad r = -1.$

15. (7 points) Given that the equation

$$\vec{x}' = \begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix} \vec{x}$$

has one solution given by

$$\vec{x}_1' = \begin{bmatrix} -2 \\ 2 \end{bmatrix} e^{3t},$$

find a second solution $\vec{x}_2$ which is linearly independent from the first.

(a) $\begin{bmatrix} -2 \\ 2 \end{bmatrix} e^{-3t}$

(b) $\begin{bmatrix} -2 \\ 2 \end{bmatrix} te^{3t}$

(c) $\left(\begin{bmatrix} -2 \\ 2 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) e^{3t}$

(d) $\left(\begin{bmatrix} -2 \\ 2 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) e^{3t}$

(e) Both (c) and (d) are solutions.

Eigenvalue is $\lambda = 3$, repeated.

Generalized eigenvector : $(A - 3I)\vec{w} = \vec{v}$.

$$\begin{bmatrix} -2 & -2 \\ 2 & 2 \end{bmatrix} \vec{w} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}.$$

Note $\vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ both work.

$x' = x - 2y$

$y' = x + 4y$ $\longrightarrow$ $x'(0) = 1 - 2 < 0$ left

$y'(0) = 1 + 4 > 0$ up

Eigenvalues : $(1-\lambda)(4-\lambda) + 2 = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3) = 0$

$\therefore$ $\lambda = 2, 3$ are positive.

16. (7 points) The planar position of a particle is given by

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

The particle starts at position $(1, 1)$; which of the following describes the motion of the particle?

(a) It initially moves up and to the right; after a long time it is very far from the origin.

(b) It initially moves up and to the left; after a long time it is very far from the origin.

(c) It initially moves up and to the right; after a long time it is very near the origin.

(d) It initially moves up and to the left; after a long time it is very near the origin.

(e) The particle moves in a closed loop around the origin.

17. (7 points) The system

$$\frac{dx}{dt} = xy - 2$$
$$\frac{dy}{dt} = x - 2y$$

has a critical point at $(2, 1)$. Classify the critical point.

(a) Spiral sink

(b) Spiral source

(c) Center

(d) Saddle

(e) None of the above

$$J = \begin{bmatrix} y & x \\ 1 & -2 \end{bmatrix}$$

$$\therefore \quad J(2,1) = \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}$$

$$(1-\lambda)(-2-\lambda) - 2 = 0$$
$$\lambda^2 - 3\lambda - 4 = 0$$
$$(\lambda - 4)(\lambda + 1) = 0 \qquad \lambda = 4, -1.$$

18. (7 points) Consider the system

Note: Can also solve by Elimination,

$$\frac{dx}{dt} = -8y$$
$$\frac{dy}{dt} = 2x$$

with initial condition $x(0) = 2, y(0) = 0$. What is the largest value that y attains?

(a) 0

(b) 1

(c) 2

(d) 4

(e) The solution is unbounded.

Trajectories are the level curves for the ellipse $\frac{x^2}{4} + y^2 = 1$,

So y ranges from -1 to 1.

Check: $\frac{d}{dt}\left(\frac{1}{4}x^2 + y^2\right)$

$= \frac{1}{2}xx' + 2yy'$

$= \frac{1}{2}x(-8y) + 2y(2x) = 0.$