Math 217 - Exam 1
17 September 2018

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Solutions

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Instructions For problems 1-10, select the correct answer from the choices provided. You do not need to show any work for these problems, and the grading is only based on the option selected.

For problems 11-12, provide a *complete* solution, showing how you arrive at the final answer. An answer without work will only receive partial credit.

The exam is worth $7 \times 10 + 15 \times 2 = 100$ points. No calculators, electronic devices, or notesheets are allowed on this exam.

1. (7 points) Given the separable equation

$$\frac{2\sqrt{x}}{y^2+1}\frac{dy}{dx} = 1,$$

$$\int \frac{dy}{y^2+1} = \int \frac{dx}{2\sqrt{x}}$$

which of the following is the general solution?

(a) $y = \tan(\sqrt{x})$

(b) $y = \tan(\sqrt{x}) + C$

(c) $y = \tan(\sqrt{x} + C)$

(d) $y = \tan(x) + C$

(e) $y^3 + y = \frac{1}{3}x^{3/2} + C$

$$\arctan y = \sqrt{x} + C$$

$$y = \tan(\sqrt{x} + C)$$

2. (7 points) For the initial value problem

$$\frac{dy}{dx} = e^{x+y}, \quad y(0) = 0,$$

$$\int e^{-y}\,dy = \int e^{x}\,dx$$

what is $y(-1)$?

(a) $-\ln(2 - e^{-1})$

(b) 0

(c) $\ln(2)$

(d) $\ln(2 + e)$

(e) Undefined

$$-e^{-y} = e^{x} + C$$

$$-1 = 1 + C \rightarrow C = -2$$

$$-e^{-y} = e^{x} - 2$$

$$y = -\ln(2 - e^{x})$$

$$\therefore y(-1) = -\ln(2-e^{-1}$$

3. (7 points) Which of the following equations are satisfied by $y = \cos(2x)$?

(a) $y'' - 4y = 0$

(b) $y' = 0$

(c) $y' + 2\sin(2x) = 0$

(d) $y'' + 4y = 0$

(e) Both (c) and (d)

$$y' = -2\sin(2x) \longrightarrow y' + 2\sin(2x) = 0$$

$$y'' = -4\cos 2x \longrightarrow y'' + 4y = 0.$$

4. (7 points) Given the linear differential equation

$$(x^2 + 1)y' + xy = \tan^{-1} x,$$

$y' + \frac{x}{x^2+1} y = \cdots$

which of the following is an integrating factor?

(a) e^x
(b) $\sqrt{x^2 + 1)}$
(c) $(x^2 + 1)^2$
(d) e^x/e^{x^2+1}
(e) $e^{\int \tan^{-1} x\, dx}$

$\mu(x) = e^{\int \frac{x}{x^2+1} dx} = e^{\frac{1}{2}\ln(x^2+1)} = \sqrt{x^2+1}.$

5. (7 points) Given the linear differential equation $xy' - 3y = x^4, y(1) = 10$, what is $y(2)$?

(a) 0
(b) 72
(c) 80
(d) 88
(e) Undefined

$y' - \frac{3}{x} y = x^3 \longrightarrow \mu = e^{\int -3/x\, dx} = e^{-3\ln x} = x^{-3}$

$x^{-3} y' - 3x^{-4} y = 1 \longrightarrow (x^{-3} y)' = 1$

$x^{-3} y = x + C \longrightarrow y = x^4 + Cx^3. \quad 10 = 1 + C \longrightarrow C = 9.$

$\therefore y(2) = 2^4 + 9 \cdot 2^3 = 88.$

6. (7 points) The acceleration of a particle at time t is $a(t) = e^{-t}$, and its initial velocity is 0. If the particle starts at position 0 at time 0, where is it at time 1?

(a) $-1/e$
(b) 0
(c) $1/e$
(d) $1 - 1/e$
(e) 1

$v(t) = v(0) + \int_0^t e^{-s} ds = 0 + (-e^{-s}|_0^t) = 1 - e^{-t}.$

$s(t) = \int_0^t 1 - e^{-s} ds = s + e^{-s}|_0^t = t + e^{-t} - 1$

$\therefore s(1) = e^{-1}.$

7. (7 points) An investment grows with continuously compounded interest; the value V of the investment at time t can be modeled by $V'(t) = kV(t)$ for a constant $k > 0$. If it takes 10 years for the investment to double, how long will it take to triple?

(a) $10k$ years
(b) $10(\ln 2)/(\ln 3)$ years
(c) 15 years
(d) $10(\ln 3)/(\ln 2)$ years
(e) $30k$ years

$V(t) = V(0) e^{kt}$

$2\cancel{V(0)} = \cancel{V(0)} e^{10k}$

$k = \frac{\ln 2}{10}.$

$3\cancel{V(0)} = \cancel{V(0)} e^{\frac{\ln 2}{10} t}$

$\ln 3 = \frac{\ln 2}{10} t$

$t = \frac{10 \ln 3}{\ln 2}$

8. (7 points) The equation $y' + y = y^3$ is a Bernoulli equation. Which of the following is an equivalent *linear* equation after an appropriate substitution?

(a) $v' + v = v^3$
(b) $v' + v = 0$
(c) $v' - 2v = -2$
(d) $v' = 3v$
(e) $v' + v = 3v$

$y^{-3} y' + y^{-2} = 1.$

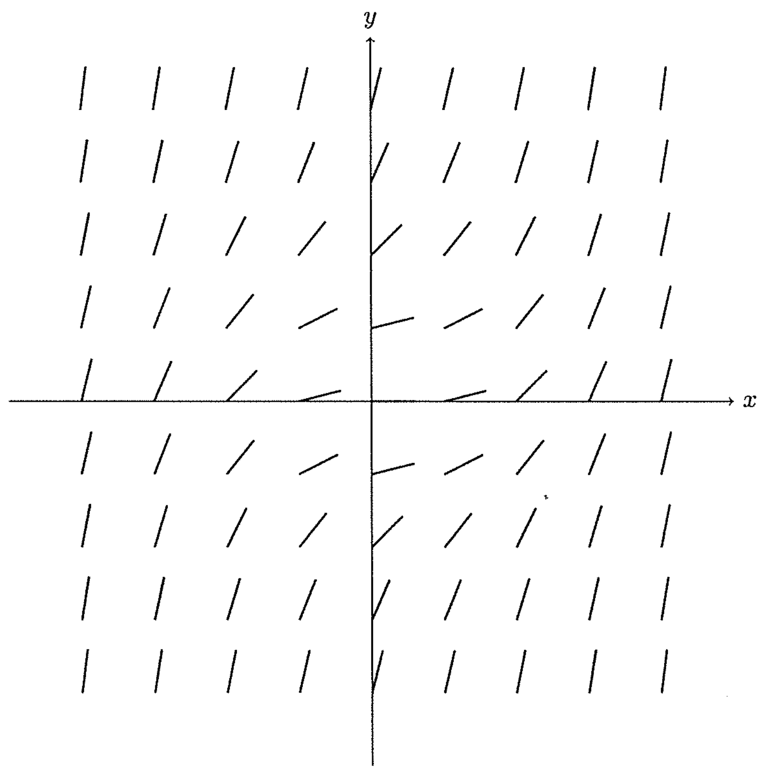
$v = y^{-2} \longrightarrow v' = -2y^{-3} y'$

$\longrightarrow y^{-3} y' = -\frac{v'}{2}$

$\therefore -\frac{v'}{2} + v = 1$

$v' - 2v = -2$

9. (7 points) Below is a slope field. Which differential equation does it correspond to?



(a) $y' = x$
(b) $y' = x + y$
(c) $y' = x^2 + y^2$
(d) $y' = x - y$
(e) $y' = x^2 - y^2$

Note y' is always ≥ 0 in this diagram.

10. (7 points) Which of the following initial value problems are guaranteed to have a unique solution on some interval containing the initial point?

(a)
$$y' = x + y, \quad y(0) = 3$$
Linear ✓

(b)
$$e^y y' = y^2 e^{x+\tan x}, \quad y(0) = 0$$
$y' = y^2 e^{-y} e^{x} \tan(x)$ has no issues. ✓

(c)
$$y'\sqrt{y} = 1, \quad y(0) = 0$$
$y' = y^{-1/2}$, $\partial_y y^{-1/2}$ is not contin at 0. Neither is $y^{-1/2}$ ...

(d)
$$y' + \frac{y}{\sin x} = 0, \quad y(\pi) = 3$$
$\frac{1}{\sin \pi}$ doesn't exist.

(e) (a) and (b) only

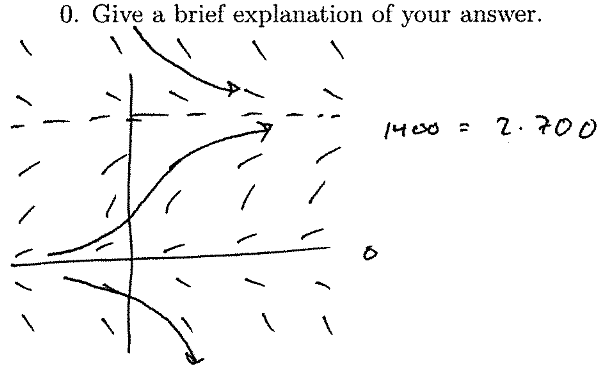
11. (**Full Answer Question**) A particular deer population can be modeled by the equation

$$\frac{dP}{dt} = 2P - \frac{1}{700}P^2.$$

(a) (10 points) Draw a slope field for this equation, and a couple of sample solutions for different initial conditions.

(b) (5 points) Without solving the equation, evaluate $\lim_{t\to\infty} P(t)$ given that $P(0) > 0$. Give a brief explanation of your answer.

Not physical!



b) The steady state solution is $P(t) = \underline{\underline{1400}}$, because $dP/dt = 0$ when $P = 1400$.

This is a logistic equation for a population with this carrying capacity, and it is evident from the slope field that

$\lim_{t\to\infty} P(t) = 1400$ for any choice of $P(0)$ that is positive.

12. (**Full Answer Question**) Consider the differential equation

$$(2xy^2 + 3x^2)dx + (2x^2y + y^3)dy = 0.$$

(a) (5 points) Verify that the equation is exact.

(b) (10 points) Give an implicit solution to the equation.

a) $M = 2xy^2 + 3x^2$ $\quad \partial_y M = 4xy$

$N = 2x^2y + y^3$ $\quad \partial_x N = 4xy$

Same, so exact.

b) $F(x, y) = C$ is the potential function.

$$F = \int M\,dx = \int (2xy^2 + 3x^2)\,dx$$

$$= x^2y^2 + x^3 + K(y)$$

$$N = \partial_y F$$

$$\cancel{2x^2y} + y^3 = \cancel{2x^2y} + K'(y)$$

$$y^3 = K'(y) \longrightarrow K(y) = \frac{1}{4}y^4$$

$$\therefore \quad \boxed{x^2y^2 + x^3 + \frac{1}{4}y^4 = C}$$