Math 217 - Practice 3

Notes to the reader: This practice exam is meant to prepare you for the second midterm exam by showing you a variety of problems that will cover many of the concepts to learn for the exam. However, that does not mean that these are the only problems to study for the test - in general, these problems may or may not be similar to what appears on the exam. You should also not use these problems or this practice as a gauge of the actual exam's difficulty. I strongly encourage you to try these problems *without* using a calculator, your notes, or any other references.

Tentatively, the actual exam will consist of 10 multipe choice questions worth 6% each, as well as 3 hand graded questions making up the remaining 40% of the exam. You should make sure that you can write clear and concise solutions to each of these problems!

1. Compute the Laplace transforms of e^{3t+1}, $\sin^2 t$, and $tu(t-1)$ directly from the definition.
2. Compute the inverse Laplace transforms of $(10s-3)/(25-s^2)$ and $2e^{-3s}/s$.
3. Using the Laplace transform, solve $x'' + 3x' + 2x = t, x(0) = 0, x'(0) = 2$; solve $x'' + x = \sin(2t), x(0) = 0, x'(0) = 0$.
4. Compute the Laplace transform of $t^2 \cos(3t)$.
5. Compute the inverse Laplace transform of $(s^2+3)/(s^2+2s+2)^2$.
6. Find the inverse Laplace transform of $\arctan 3/(s+2)$.
7. Updated: Find a solution to $tx'' - 2x' + tx = 0$ under the assumption $x(0) = 0$.
8. Write a convolution formula for the solution to $x'' + 4x' + 13x = f(t)$.
9. Solve $x'' + 5x' + 4x = f(t)$ where $f(t) = 1$ for $0 \le t < 2$ and 0 otherwise.
10. Solve $x'' + 9x = \delta(t - 3\pi) + \cos(3t)$ with $x(0) = 0, x'(0) = 0$.
11. Find a power series solution to $(x-2)y' + y = 0$ and then use the coefficients to find the radius of convergence. Repeat for $y'' + y = x$.
12. Find the guaranteed radius of convergence for $(x^2+3)y'' - 7xy' + 16y = 0$ for a series solution centered at $x = 0$, and then find the recurrence relation. Then find the guaranteed radius of convergence if we center the series at $x = 10$ instead.
13. Classify 0 as an ordinary point, a regular singular point, or an irregular singular point for $xy'' + x^2y' + (e^x - 1)y = 0$.
14. Find two linearly independent Frobenius series solutions for $4xy'' + 2y' + y = 0$.

Our standard algorithm for computing series solutions is:

1. Identify whether the center is an ordinary (good), regular singular (ok), or irregular singular point (bad). If ordinary, go with $y = \sum_{n=0}^{\infty} c_n x^n$ and identify the radius of convergence from the closest singular point. If regular, go with $x^r \sum_{n=0}^{\infty} c_n x^n$ with r a root of the indicial polynomial (Frobenius solution).
2. Compute y' and y'' and bring every term into a standard form, with the same starting index and the same power. You may need to shift indices or peel off the first few terms of a piece.
3. Find a recurrence relation for the coefficients.
4. Write out enough coefficients to identify the pattern, and solve the recurrence.
5. Write the solution(s) as explicitly as possible, and then bring in the initial conditions as necessary.

The following table will be given on the exam, and you may use it without proof:

1. $\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\, dt$ provided that the integral converges
2. $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$ and $\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$
3. $\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s)$
4. $\mathcal{L}\{(f * g)(t)\} = F(s)G(s)$ holds for the convolution of two functions
5. $\mathcal{L}\{tf(t)\} = -F'(s)$
6. $\mathcal{L}\{t^n\} = n!/s^{n+1}$
7. $\mathcal{L}\{e^{at}\} = 1/(s-a)$ and $\mathcal{L}\{t^n e^{at}\} = n!/(s-a)^{n+1}$
8. $\mathcal{L}\{\sin(kt)\} = k/(s^2+k^2)$ and $\mathcal{L}\{\cos(kt)\} = s/(s^2+k^2)$
9. $\mathcal{L}\{u(t-a)\} = e^{-as}/s$ and $\mathcal{L}\{\delta(t-a)\} = e^{-as}$

Answers.

1. The transforms are $e/(s-3)$, $2/(s^3+4s)$, and $e^{-s}(1/s+1/s^2)$, respectively.
2. The inverse transforms are $(-47/10)e^{5t}+(-53/10)e^{-5t}$ and $2u(t-3)$, respectively.
3. The solutions are $\frac{1}{4}(2t-9e^{-2t}+12e^{-t}-3)$ and $\frac{2}{3}\sin t-\frac{1}{3}\sin 2t$, respectively.
4. The transform can be simplified as $2s(s^2-27)/(s^2+9)^3$.
5. The inverse transform can be written as $-\frac{1}{2}e^{-t}(2t-5)\sin 5t-\frac{3}{2}e^{-t}\cos t$.
6. The inverse transform is $e^{-2t}\sin(3t)/t$.
7. One solution is $t\cos t-\sin t$.
8. The solution is $x(t)=\int_0^t \frac{1}{3}e^{-2\tau}\sin(3\tau)f(t-\tau)$.
9. The general solution is

$$x(t)=c_1e^{-4t}+c_2e^{-t}+\begin{cases}\frac{1}{4} & t\le 2\\ \frac{1}{12}e^{2-4t}(-e^6+4e^{3t}) & t>2\end{cases}$$

10. The solution is $x(t)=\frac{1}{6}(t-2u(t-3\pi)sin(3t))$.
11. The first solution is

$$y(t)=\frac{c}{2-x}=\frac{c/2}{1-x/2}=\frac{c}{2}\sum_{n=0}^{\infty}(x/2)^n$$

and the radius of convergence is 2. The second solution is

$$x+\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}+\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

and the radius is $+\infty$.

12. The only singular points are at $\pm\sqrt{3}i$, so the radius is $\sqrt{3}$. The recurrence relation is

$$n(n-1)c_n+3(n+2)(n+1)c_{n+2}-7nc_n+16c_n=0.$$

For the second point, the radius is the distance from $(10,0)$ to $(0,\pm\sqrt{3})$ in the plane (draw the picture!!) and the radius is $\sqrt{10^2+\sqrt{3}^2}=\sqrt{109}$.

13. It's an ordinary point if we replace $(e^x-1)/x$ with 1 when $x=0$.
14. The general solution is

$$y=A\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{n+1/2}+B\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^n=A\sin\sqrt{x}+B\cos\sqrt{x}.$$