Laplace & Partial Fractions

Recall: We have a library of rules and functions to go between $f(t)$ and $F(s) = \mathcal{L}\{f(t)\}$ or $F(s)$ and $\mathcal{L}^{-1}\{F(s)\}$. We would like a way to reduce rational functions to simpler functions in our library

Ex: Compute $\mathcal{L}^{-1}\left\{\frac{s^2+1}{s(s+2)(s+3)}\right\}$:

use method of partial fractions from Calc II:

$$\frac{s^2+1}{s(s+2)(s+3)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+3} \Rightarrow$$

$$A(s+2)(s+3) + B(s)(s+3) + C(s)(s+2) = s^2+1$$

If $s=0$ $\quad 6A = 1 \quad A = \frac{1}{6}$

If $s=-2$ $\quad -2B = 5 \quad B = -\frac{5}{2}$ $\quad \Rightarrow \frac{s^2+1}{s(s+2)(s+3)} = \frac{1}{6}\left(\frac{1}{s}\right) - \frac{5}{2}\left(\frac{1}{s+2}\right) + \frac{10}{3}\left(\frac{1}{s+3}\right)$

If $s=-3$ $\quad 3C = 10 \quad C = \frac{10}{3}$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{s^2+1}{s(s+2)(s+3)}\right\} = \frac{1}{6}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{5}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \frac{10}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$$

$$= \frac{1}{6} - \frac{5}{2}e^{-2t} + \frac{10}{3}e^{-3t}$$

Ex: Solve the DE $y'' + 4y' + 4y = t^2$ $\quad y(0) = y'(0) = 0$

$$\mathcal{L}\{y''(t) + 4y'(t) + 4y\} = \mathcal{L}\{t^2\}$$

$$\Rightarrow [s^2Y(s) - \cancel{sy(0)} - \cancel{y'(0)}] + 4[sY(s) - \cancel{y(0)}] + 4[Y(s)] = \frac{2}{s^3}$$

$$\Rightarrow [s^2 + 4s + 4]Y(s) = \frac{2}{s^3}$$

$$\Rightarrow Y(s) = \frac{2}{s^3(s^2+4s+4)} = \frac{2}{s^3(s+2)^2}$$

$$\Rightarrow Y(s) = \underbrace{\frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3}}_{\text{will associate to particular}} + \underbrace{\frac{D}{s+2} + \frac{E}{(s+2)^2}}_{\text{will associate to } e^{-2t},\ te^{-2t}}$$

Upshot: When we take the inverse Laplace transform, we essentially repeat the process we used with finding roots of characteristic polynomials and using particular solutions. The coefficients A, B, C, D, E associate to using IVP, homogeneous eq to find coefficients previously.

Recall: Every polynomial can be factored into a product of linear eq $(x+a)$ and quadratics with complex roots (x^2+bx+c) times a constant

- this means we must only deal with cases $\frac{A}{x+a}$ and $\frac{Bx+C}{x^2+bx+c}$
- Remember $\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t)$ where $\mathcal{L}\{f(t)\} = F(s)$

Ex: $\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+4}\right\} = e^t\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}e^t\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = \frac{1}{2}e^t\sin 2t$

Ex: $\mathcal{L}^{-1}\left\{\frac{2s^2+10s}{(s^2-2s+5)(s+1)}\right\}$ we have s^2-2s+5 is irreducible $b^2-4ac = 4-20<0$

$$\Rightarrow \frac{2s^2+10s}{(s^2-2s+5)(s+1)} = \frac{As+B}{s^2-2s+5} + \frac{C}{s+1}$$

$\Rightarrow [As+B][s+1] + C[s^2-2s+5] = 2s^2+10s$ ← trick can't be used as a whole

$As^2 + As + Bs + B + Cs^2 - 2Cs + 5C = 2s^2+10s$

$\Rightarrow s^2: A+C=2$
$s: A+B-2C=10$
$1: B+5C=0$

$B-3C=8$
$B+5C=0$

$-8C=8 \quad C=-1$

$\Rightarrow B-5=0 \Rightarrow B=5$

$\Rightarrow A-1=2 \Rightarrow A=3$

$\Rightarrow = \frac{3s+5}{s^2-2s+5} + \frac{-1}{s+1}$ must complete square on first term

$s^2-2s+5 = s^2-2s+1+4$
$= (s-1)^2+4$

$$= \frac{3(s-1)+8}{(s-1)^2+4} - \frac{1}{s+1}$$

$\Rightarrow 3\mathcal{L}^{-1}\left\{\frac{(s-1)}{(s-1)^2+4}\right\} + 4\mathcal{L}^{-1}\left\{\frac{2}{(s-1)^2+4}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = 3e^t\cos(2t) + 4e^t\sin(2t) - e^{-t}$

Ex: Solve $x'' + 6x' + 34x = 30\sin 2t \quad x(0) = x'(0) = 0$

$$\mathcal{L}\{x'' + 6x' + 34x\} = \mathcal{L}\{30\sin 2t\}$$

$$\Rightarrow [s^2X(s) - \cancel{sx(0)} - \cancel{x'(0)}] + 6[sX(s) - \cancel{x(0)}] + 34[X(s)] = 30\left[\frac{2}{s^2+4}\right]$$

$$\Rightarrow X(s)[s^2+6s+34] = \frac{60}{s^2+4} \quad \Rightarrow X(s) = \frac{60}{(s^2+6s+34)(s^2+4)}$$

$b^2 - 4ac = 36 - 4\cdot 34 < 0$ no real roots

$$\frac{60}{(s^2+6s+34)(s^2+4)} = \frac{As+B}{s^2+6s+34} + \frac{Cs+D}{s^2+4}$$

$$\Rightarrow (As+B)(s^2+4) + (Cs+D)(s^2+6s+34) = 60$$

$$\left.\begin{array}{l} s^3: A + C = 0 \\ s^2: B + D + 6C = 0 \\ s: 4A + 34C + 6D = 0 \\ 1: 4B + 34D = 60 \end{array}\right] \Rightarrow \begin{array}{l} C = -A \\ B + D - 6A = 0 \\ -30A + 6D = 0 \Rightarrow 5A = D \\ 4B + 34D = 60 \end{array}$$

$$\Rightarrow \begin{array}{l} B - A = 0 \\ 4B + 170A = 60 \end{array} \quad 174A = 60 \quad A = \frac{10}{29} \quad \Rightarrow C = \frac{-10}{29}$$

$$B = \frac{10}{29} \quad D = \frac{50}{29}$$

$$= \frac{10}{29}\cdot\frac{s+1}{s^2+6s+34} - \frac{10}{29}\cdot\frac{s-5}{s^2+4} =$$

$$= \frac{10}{29}\left[\frac{s+3}{(s+3)^2+25}\right] - \frac{10}{29}\cdot\frac{2}{5}\left[\frac{5}{(s+3)^2+25}\right] - \frac{10}{29}\left[\frac{s}{s^2+4}\right] + \frac{10}{29}\cdot\frac{5}{2}\left[\frac{2}{s^2+4}\right]$$

$$\Rightarrow \mathcal{L}^{-1}(X(s)) = \frac{10}{29}e^{-3t}\cos(5t) - \frac{2\cdot 10}{29\cdot 5}e^{-3t}\sin(5t)$$

$$- \frac{10}{29}\cos(2t) + \frac{10\cdot 5}{29\cdot 2}\sin(2t) \quad = x(t)$$

Note: this corresponds to the type of solution we'd have for springs and could solve with undetermined coeff.

similar problems 27 - 33