Friday 10/19: Convolutions & Calculus.

The convolution of f & g is

$$(f * g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau.$$

The key use is $\mathcal{L}\{f * g\} = \mathcal{L}\{f\}\,\mathcal{L}\{g\}$.

So <u>Laplace turns $*$ into multiplication.</u>

Calculus:
$$\left.\begin{aligned} \mathcal{L}\{-t f(t)\} &= F'(s). \\ \mathcal{L}\left\{\frac{f(t)}{t}\right\} &= \int_s^\infty F(\sigma)\, d\sigma. \end{aligned}\right\} \quad \text{Compare to } \mathcal{L}\{f'\} = sF(s) - f(0)!$$

$\hookrightarrow$ if $\left|\lim_{t \to 0^+} \frac{f(t)}{t}\right| < \infty$. (exists & is finite).

Ex. $\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{\sigma^2+1}\, d\sigma = \frac{\pi}{2} - \arctan s = \arctan \frac{1}{s}.$

Ex. $tx'' - 2x' + tx = 0, \quad x(0) = 0$

$\hookrightarrow$ use $\mathcal{L}\{tx\} = -X'(s)$

$\mathcal{L}\{tx''\} = -(s^2X(s))' = -s^2X'(s) - 2sX(s)$

Get a first-order separable equation

$$-s^2X' - 2sX - 2sX - X' = 0$$

$$\therefore \quad X(s) = \frac{1}{(s^2+1)^2}$$

$$\therefore \quad \boxed{x(t) = \sin t - t\cos t.}$$