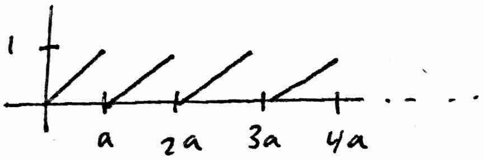
Transforms of periodic functions

Recall: A periodic function is one that is essentially "copied" at an interval and "pasted" for the rest of the graph.

Thm: Let $f(t)$ be periodic with period p (i.e $f(t+p)=f(t)$ for all t) and be piecewise continuous for $t \geq 0$. Then $F(s)=\mathcal{L}\{f(t)\}$ exists for $s>0$ and is given by:

$$F(s) = \frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)\, dt$$

Ex: compute the Laplace transform of the sawtooth:



i.e. $f(t)=\frac{1}{a}t$ on $[0,a]$, period a

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-as}} \cdot \int_0^a e^{-st} \cdot [\tfrac{1}{a}t]\, dt \qquad \frac{1}{a}\int_0^a t e^{-st}\, dt$$

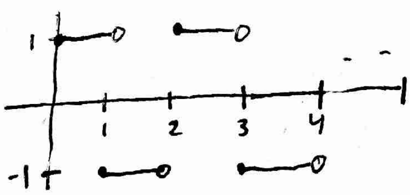
$u=t \qquad dv = e^{-st}$

$du = dt \qquad v = -\frac{1}{s}e^{-st}$

$$= \frac{1}{a}\Big[-\frac{t}{s}e^{-st} - \int -\frac{1}{s}e^{-st}\Big]_0^a = \frac{1}{a}\Big[-\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st}\Big]_0^a = \frac{1}{a}\Big[-\frac{a}{s}e^{-as} - \frac{1}{s^2}e^{-as} + \frac{1}{s^2}\Big]$$

$$= \frac{1}{a}\left[\frac{-ase^{-as} + (1-e^{-as})}{s^2}\right] \times \frac{1}{1-e^{-as}} = \frac{-e^{-as}}{s^2(1-e^{-as})} + \frac{1}{as}$$

Ex: Solve $y'' =$



$= f(t)$ 2-periodic

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-2s}} \int_0^2 e^{-st} f(t) = \frac{1}{1-e^{-2s}}\Big[\int_0^1 e^{-st} + \int_1^2 -e^{-st}\Big]$$

$$= \frac{1}{1-e^{-2s}}\Big[\Big[-\frac{1}{s}e^{-st}\Big]_0^1 + \Big[\frac{1}{s}e^{-st}\Big]_1^2\Big] = \frac{1}{1-e^{-2s}}\Big(-\frac{1}{s}e^{-s} + \frac{1}{s} + \frac{1}{s}e^{-2s} - \frac{1}{s}e^{-s}\Big)$$

$$= \frac{1}{1-e^{-2s}} \cdot \frac{1}{s} \cdot (e^{-s}-1)^2 = \frac{(1-e^{-s})^{\cancel{2}}}{s(1+e^{-s})\cancel{(1-e^{-s})}} = \frac{(1-e^{-s})}{s(1+e^{-s})}$$

we'll assume $y'(0) = y(0) = 0$ to find a solution

$\mathcal{L}\{y''\} = s^2 Y(s) + \cancel{s y(0)} + \cancel{y'(0)}$

$\Rightarrow s^2 Y(s) = \dfrac{(1-e^{-s})}{s(1+e^{-s})} \Rightarrow Y(s) = \dfrac{(1-e^{-s})}{s^3(1+e^{-s})}$ and since $s>0$,

we must have $e^{-s} < 1 \Rightarrow$ we can use the summation

formula for $\dfrac{1}{1+e^{-s}} \Rightarrow \dfrac{(1-e^{-s})}{s^3(1+e^{-s})} = \dfrac{(1-e^{-s})}{s^3} \sum_{n=0}^{\infty} (-1)^n e^{-ns}$

$= \frac{1}{s^3} \sum_{n=0}^{\infty} (-1)^n [e^{-ns} - e^{-(n+1)s}] = \frac{1}{s^3}([1 - e^{-s}] - [e^{-s} - e^{-2s}] + [e^{-2s} - e^{-3s}])$

$= \frac{1}{s^3} + \sum_{n=1}^{\infty} (-1)^n \cdot 2 \cdot e^{-ns} \Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s^3} + \sum_{n=1}^{\infty} (-1)^n \cdot 2 \cdot e^{-ns}\right\} = \frac{t^2}{2} + \sum_{n=1}^{\infty} (-1)^n \mathcal{L}^{-1}\left\{\frac{2e^{-ns}}{s^3}\right\}$

$= \frac{t^2}{2} + \sum_{n=1}^{\infty} (-1)^n u(t-n)(t-n)^2$ which is a finite sum for

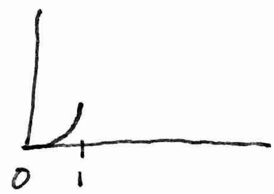
each finite t value: if $0 \le t < 1$ then $x(t) = \frac{t^2}{2}$

if $m \ge 1$ and $m \le t < m+1$ then $x(t) = \frac{t^2}{2} + \sum_{n=1}^{m} (-1)^n (t-n)^2$

we can think about what this looks like

iteratively:

For $0 \le t < 1$ the graph is $\frac{t^2}{2}$

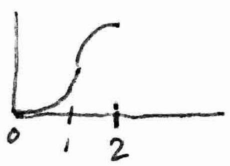


for $1 \le t < 2$ the function

becomes $\frac{t^2}{2} - (t-1)^2$ which is $\frac{t^2}{2}$ at 1 so it picks

up where the last one left off and becomes

$-\frac{t^2}{2} + 2t - 1$



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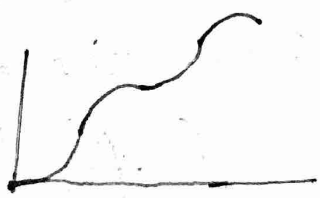

quadratic pieces

Rmk: In terms of motion, force is constant but alternates direction.

Ex: working with δ. Solve $x'' + 4x' + 4x = 1 + 3\delta(t-7)$

$x(0) = 0 \quad x'(0) = 0$

$\mathcal{L}\{x'' + 4x' + 4x\} =$ since $\begin{matrix} x(0)=0 \\ x'(0)=0 \end{matrix}$ $s^2 X(s) + 4s X(s) + 4X(s)$

$\mathcal{L}\{1 + 3\delta(t-7)\} = \frac{1}{s} + 3e^{-7s}$

$\Rightarrow (s^2 + 4s + 4)X(s) = \frac{1}{s} + 3e^{-7s} \Rightarrow X(s) = \frac{1}{s(s+2)^2} + \frac{3e^{-7s}}{(s+2)^2}$

$\mathcal{L}^{-1}\left\{\frac{1}{s(s+2)^2}\right\} = \int_0^t \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\} d\tau = \int_0^t e^{-2\tau}\tau \, d\tau$

$u = \tau \quad dv = e^{-2\tau}$
$du = d\tau \quad v = -\frac{1}{2}e^{-2\tau}$ $= \Big[-\frac{1}{2}\tau e^{-2\tau} - \int e^{-2\tau} d\tau \Big]_0^t$

$= \Big[-\frac{1}{2}\tau e^{-2\tau} + \frac{1}{2}e^{-2\tau}\Big]_0^t = -\frac{1}{2}te^{-2t} + \frac{1}{2}e^{-2t} - \frac{1}{2}$

$\mathcal{L}^{-1}\left\{\frac{3e^{-7s}}{(s+2)^2}\right\} = 3u(t-7)e^{-2(t-7)}(t-7)$

$\Rightarrow x(t) = -\frac{1}{2}te^{-2t} + \frac{1}{2}e^{-2t} - \frac{1}{2} + 3u(t-7)e^{-2(t-7)}(t-7)$