Wednesday 11/7 – Frobenius series.

If we have $x^2 y'' + x\,p(x)\,y' + q(x)\,y = 0$ then $x = 0$ is potentially a singular point. So instead of a Taylor series, we use a Frobenius series $\sum_{n=0}^{\infty} c_n x^{n+r}$. The exponents $r$ solve the <u>indicial polynomial</u> $r(r-1) + p_0 r + q_0 = 0$.

<u>Ex.</u> $xy'' + 2y' + xy = 0$, or $x^2 y'' + 2x y' + x^2 y = 0$.

1) Find candidates. $p = 2$, $q = x^2 \rightsquigarrow r(r-1) + 2r + 0 = 0$

$\longrightarrow r = 0, -1$.

2) Find recurrence. Let's just look at $r = -1$:

$y = \sum_{n=0}^{\infty} c_n x^{n-1} \longrightarrow xy' = \sum_{n=0}^{\infty} (n-1) c_n x^{n-1}$

$\longrightarrow x^2 y'' = \sum_{n=0}^{\infty} (n-1)(n-2)\, c_n x^{n-1}$.

So get
$$\sum_{n=0}^{\infty} c_n (n-1)(n\underline{-2}) x^{n-1} + \underline{\sum_{n=0}^{\infty} 2(n-1)\, c_n x^{n-1}} + \sum_{n=0}^{\infty} c_n x^2 \cdot x^{n-1} = 0$$

cancel!

$$\therefore \sum_{n=2}^{\infty} c_n\, n(n-1) x^{n-1} + \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

$$\therefore \sum_{n=0}^{\infty} \left[ c_{n+2} (n+2)(n+1) + c_n \right] x^{n+1} = 0$$

$$\therefore c_{n+2} = -\frac{c_n}{(n+2)(n+1)}.$$

3) Solution for evens: $c_{2n} = \frac{(-1)^n c_0}{(2n)!}$

Odds are similar.

So we get a solution ($c_0 = 1$, $c_1 = 0$) of
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n-1} = \boxed{\frac{\cos x}{x}}$$