Wednesday 9/12. Continuing substitutions, exact equations.

A homogeneous equation is $y' = F(y/x)$, and we can think of y & x as being coupled as a single variable $\longrightarrow$ Let $v = y/x$. Then

$$y' = (xv)' = xv' + v.$$

Ex. $xy^2y' = x^3 + y^3 \longrightarrow y' = \left(\frac{x}{y}\right)^2 + \frac{y}{x}$ is homogeneous. It becomes

$$xv' + v = \frac{1}{v^2} + v \longrightarrow xv' = \frac{1}{v^2}$$

which is separable.

Exact equations are of the form $M + N\frac{dy}{dx} = 0$ with $\partial_y M = \partial_x N$. They come from potential functions $F(x, y(x)) = C \longrightarrow \frac{d}{dx} F(x, y(x)) = \frac{d}{dx} C$

$$\longrightarrow \underset{\substack{\downarrow \\ M}}{\frac{\partial F}{\partial x}} + \underset{\substack{\downarrow \\ N.}}{\frac{\partial F}{\partial y}} \frac{dy}{dx} = 0$$

Theory says $\partial_y M = \partial_x N$.

Ex. $\underbrace{(\cos x + \ln y)}_{M}\,dx + \underbrace{(x/y + e^y)}_{N}\,dy = 0$ is exact: $M_y = \frac{1}{y} = N_x$.

$$F = \int M\,dx = \sin x + x\ln y + K(y)$$

$$\partial_y F = N \longrightarrow x/y + K'(y) = x/y + e^y \longrightarrow K(y) = e^y.$$

So we get $\boxed{\sin x + x\ln y + e^y = \text{Constant.}}$