Homework 10
Math 217
Due: 7 December 2018 by 11:59 PM

Instructions: Write your solutions to the following problems and submit them on Crowdmark by the deadline. You are encouraged to work in groups or consult with each other on the problems, but the work submitted must be your own and must be written up by you.

(1) Find the (real-valued) solution to the system

\[
\begin{align*}
x' &= 5x + 5y + 2z \\
y' &= -6x - 6y - 5z \\
z' &= 6x + 6y + 5z
\end{align*}
\]

with initial conditions \(x(0) = 3, y(0) = 2, z(0) = 1\).

Solution. The characteristic equation of the underlying matrix is

\[
0 = \det \begin{bmatrix} 5 - \lambda & 5 & 2 \\ -6 & -6 - \lambda & -5 \\ 6 & 6 & 5 - \lambda \end{bmatrix}
\]

\[
= (5 - \lambda)((-6 - \lambda)(5 - \lambda) + 30) - 5(-6(5 - \lambda) + 30) + 2(-36 - 6(-6 - \lambda))
\]

\[
= (5 - \lambda)(\lambda + \lambda^2) - 5(6\lambda) + 2(6\lambda)
\]

\[
= -\lambda^3 + 4\lambda^2 - 13\lambda
\]

\[
= -\lambda(\lambda^2 - 4\lambda + 13)
\]

which leads us to eigenvalues \(\lambda_1 = 0, \lambda_2 = 2 + 3i, \) and \(\lambda_3 = 2 - 3i\). Now we need the eigenvectors:

- For \(\lambda = 0\), we need to find a vector \(\vec{v} = [a, b, c]^T\) for which

\[
\begin{bmatrix} 5 & 5 & 2 \\ -6 & -6 & -5 \\ 6 & 6 & 5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]

which is equivalent to the system

\[
\begin{align*}
5a + 5b + 2c &= 0 \\
-6a - 6b - 5c &= 0 \\
6a + 6b + 5c &= 0
\end{align*}
\]

Note that we only need one non-zero vector of this form to construct a solution, and that we only have two equations to determine three variables. Let’s try looking for a solution with \(a = 1\) (since we want a non-zero vector; if this doesn’t work, try with \(b = 1\) or \(c = 1\) in turn). This leads to

\[
5b + 2c = -5 \\
6b + 5c = -6
\]

which can be solved. In particular, \(b = -1, c = 0\) gives a solution. So we can take as our first eigenvector \([1, -1, 0]^T\) or any multiple thereof.
• For the eigenvalue $\lambda = 2 + 3i$, we need to find a vector $\vec{v} = [a, b, c]^T$ for which

$$
\begin{bmatrix}
3 - 3i & 5 & 2 \\
-6 & -8 - 3i & -5 \\
6 & 6 & 3 - 3i
\end{bmatrix}
\begin{bmatrix}
a \\
b \\
c
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}.
$$

Set $c = 1$ and solve the corresponding system of (complex) linear equations. In particular, we would have (from the first two rows)

$$(3 - 3i)a + 5b = -2 \quad -6a + (-8 - 3i)b = 5$$

Multiplying the first equation by $1 + i$ gives

$$6a + (5 + 5i)b = -2 - 2i$$

and adding it to the second equation gives us

$$(-3 + 2i)b = 3 - 2i$$

leading to $b = -1$. Finally, $a = -3/(3 - 3i) = (1 + i)/2$. Scaling by 2 to clear the fraction gives us the choice $[1 + i, -2, 2]^T$. Alternatively, if you used a computer to solve this, that’s fine.

• The eigenvector is the conjugate of the previous one, so $[1 - i, -2, 2]^T$.

Superimposing, our general solution is

$$c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 + i \\ -2 \\ 2 \end{bmatrix} e^{(2+3i)t} + c_3 \begin{bmatrix} 1 - i \\ -2 \\ 2 \end{bmatrix} e^{(2-3i)t}.$$ 

The middle term can be written as

$$e^{2t} \left( \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) (\cos(3t) + i \sin(3t)) = e^{2t} \begin{bmatrix} \cos(3t) - \sin(3t) \\ -2\cos(3t) \\ 2\cos(3t) \end{bmatrix} + ie^{2t} \begin{bmatrix} \sin(3t) + \cos(3t) \\ -2\sin(3t) \\ 2\sin(3t) \end{bmatrix}$$

so that we can write a real-valued general solution as

$$A \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + Be^{2t} \begin{bmatrix} \cos(3t) - \sin(3t) \\ -2\cos(3t) \\ 2\cos(3t) \end{bmatrix} + Ce^{2t} \begin{bmatrix} \sin(3t) + \cos(3t) \\ -2\sin(3t) \\ 2\sin(3t) \end{bmatrix}.$$ 

Finally, we need the constants. Using that $x(0) = 3, y(0) = 2, z(0) = 1$, we have

$$A + B + C = 3$$
$$-A - 2B = 2$$
$$2B = 1$$

immediately implying $B = 1/2, A = -3,$ and $C = 11/2$. Thus our solution is

$$
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} = -3 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \frac{1}{2} e^{2t} \begin{bmatrix} \cos(3t) - \sin(3t) \\ -2\cos(3t) \\ 2\cos(3t) \end{bmatrix} + \frac{11}{2} e^{2t} \begin{bmatrix} \sin(3t) + \cos(3t) \\ -2\sin(3t) \\ 2\sin(3t) \end{bmatrix}.
$$
Find the general solution of
\[ \vec{x}' = \begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix} \vec{x}. \]

**Solution.** The characteristic polynomial of the matrix is
\[(3 - \lambda)(5 - \lambda) + 1 = \lambda^2 - 8\lambda + 16 = (\lambda - 4)^2.\]
We have a repeated eigenvalue of \(\lambda = 4\). The unique eigenvector satisfies
\[ \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \vec{v} = \vec{0} \implies \vec{v} = \begin{bmatrix} a \\ -a \end{bmatrix} \]
for a choice of \(a \neq 0\). For simplicity, choose \(a = 1\).
In order to construct a second solution, we need to find a generalized eigenvector, i.e. a vector \(\vec{w}\) for which \((A - \lambda I)\vec{w} = \vec{v}\). To this end, take
\[ \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \vec{w} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \]
Many vectors \(w\) will work, such as \([-1, 0]^T\) and \([0, -1]^T\). Choosing the first one, we can build our general solution; we have
\[ \vec{x} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{4t} + c_2 \left( \begin{bmatrix} 1 \\ -1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) e^{4t}. \]

Find the critical points of the system
\[ \frac{dx}{dt} = y - 1 \]
\[ \frac{dy}{dt} = x^2 - y. \]
Classify each critical point as stable/unstable/saddle, and determine whether the trajectories have any spiraling. Draw a rough phase portrait, making sure to identify the critical points and with enough detail to show the long-term behavior of the system.

**Solution.** The critical points are \((\pm 1, 1)\). The Jacobian matrix is given by
\[ J = \begin{bmatrix} 0 & 1 \\ 2x & -1 \end{bmatrix} \]
and so we need to compute its eigenvalues at each critical point.

- **At \((1, 1)\):** The Jacobian is
\[ \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \]
and has characteristic equation
\[ 0 = -\lambda(-1 - \lambda) - 2 = \lambda^2 + \lambda - 2 = (\lambda + 2)(\lambda - 1). \]
Hence one of the roots is positive and one is negative, so we have a **saddle**.

- **At \((-1, 1)\):** The Jacobian is
\[ \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix} \]
and has characteristic equation
\[ 0 = -\lambda(-1 - \lambda) + 2 = \lambda^2 + \lambda + 2 \]
which has roots
\[ \lambda = \frac{-1 \pm \sqrt{-7}}{2}. \]

Since the real part is negative and we have non-zero imaginary part, we get a **spiral sink**, or a stable spiraling point.

A nicely drawn phase portrait can be found in the back of the textbook; this is problem 30 of section 6.2.

(4) Carry out a careful analysis of the following two-population system. Draw a phase portrait (in the domain \( x, y \geq 0 \)), identify any equilibrium point(s) as stable/unstable/saddles, and determine under what circumstances the populations will coexist indefinitely.

\[
\begin{align*}
\frac{dx}{dt} &= 2xy - 16x \\
\frac{dy}{dt} &= 4y - xy.
\end{align*}
\]

**Solution.** This was done in class on Wednesday as an example. Since we have

\[
\begin{align*}
x' &= 2x(y - 8) \\
y' &= y(4 - x)
\end{align*}
\]

we have critical points at \((0, 0)\) and \((4, 8)\). The Jacobian is given by

\[
J = \begin{bmatrix}
2y - 16 & 2x \\
-y & 4 - x
\end{bmatrix}
\]

and so we need to compute its eigenvalues at each critical point.

- **At \((0, 0)\):** The Jacobian is

  \[
  \begin{bmatrix}
  -16 & 0 \\
  0 & 4
  \end{bmatrix}
  \]

  and has eigenvalues \(\lambda_1 = -16 < 0\) and \(\lambda_2 = 4 > 0\). This corresponds to a saddle.

- **At \((4, 8)\):** The Jacobian is

  \[
  \begin{bmatrix}
  0 & 8 \\
  -8 & 0
  \end{bmatrix}
  \]

  and has eigenvalues \(\pm 8i\). This leads to an apparent center.

A nicely drawn phase portrait can be found in the back of the textbook; this is problem 28 of section 6.3. In particular, as long as both species populations are non-zero, they will coexist indefinitely (and typically oscillated back and forth between which is larger). We can interpret this as a predator-prey model where the species \(x\) is predator (its growth increases with \(y\), but exhibits natural decay) and \(y\) is prey (its growth decreases with \(x\), but exhibits natural growth).