Homework 1

Math 217

Due: 5 September 2018 by 11:00 PM

Instructions: Write your solutions to the following problems and submit them on Crowdmark by the deadline. You are encouraged to work in groups or consult with each other on the problems, but the work submitted must be your own and must be written up by you.

(1) (Variant of 1.1 #15) Determine all values of \( r \) for which \( y(x) = e^{rx} \) is a solution to the equation

\[
y'' - 2y' - 15y = 0.
\]

Solution

If \( y(x) = e^{rx} \), then \( y'(x) = re^{rx} \) and \( y''(x) = r^2e^{rx} \). Substituting this into the equation, we find

\[
r^2e^{rx} - 2re^{rx} - 15e^{rx} = 0 \implies (r^2 - 2r - 15)e^{rx} = 0
\]

\[
r^2 - 2r - 15 = 0
\]

\[
(r - 5)(r + 3) = 0
\]

so that \( r = 5 \) or \( r = -3 \). We can then check that both of these are solutions, since \((e^{5x})'' - 2(e^{5x})' - 15e^{5x} = 25e^{5x} - 10e^{5x} - 15e^{5x} = 0\) and \((e^{-3x})'' - 2(e^{-3x})' - 15e^{-3x} = 9e^{-3x} + 6e^{-3x} - 15e^{-3x} = 0\). Note that the first part (concluding that \( r \in \{-3, 5\} \) constrains the form of the solutions, but does not prove that either candidate is a solution!

(2) (1.2 #10) Solve the initial value problem

\[
\frac{dy}{dx} = xe^{-x}, \quad y(0) = 1.
\]

Solution

This equation is separable:

\[
dy = xe^{-x} \, dx
\]

\[
y(x) = \int xe^{-x} \, dx
\]

\[
y(x) = -xe^{-x} - \int (-e^{-x}) \, dx
\]

\[
y(x) = -xe^{-x} - e^{-x} + C
\]

after integrating by parts. Using the information that \( y(0) = 1 \), we have

\[
1 = 0 - e^0 + C \implies C = 2
\]

and our final answer is

\[
y(x) = -xe^{-x} - e^{-x} + 2.
\]

Note: An alternative solution is to write

\[
y(x) = 1 + \int_0^x te^{-t} \, dt
\]

and integrate. This doesn’t require computing \( C \) at the end, since it’s a definite integral.
(3) (1.4 #24) Find an explicit solution to the initial value problem

\[
\frac{dy}{dx} = y \cot x, \quad y\left(\frac{\pi}{2}\right) = \frac{\pi}{2}.
\]

**Solution** This is also separable. We have

\[
\frac{dy}{y} = \cot x \, dx
\]

\[
\int \frac{dy}{y} = \int \frac{\cos x}{\sin x} \, dx
\]

\[
\ln |y| = \ln |\sin x| + C
\]

Now given the initial condition, note that both \(y\) and \(\sin x\) are positive. So we can drop the absolute values to get

\[
\ln y = \ln \sin x + C \implies y = e^C \sin x.
\]

Finally, the initial condition tells us

\[
\frac{\pi}{2} = e^C \sin \frac{\pi}{2} \implies e^C = \frac{\pi}{2}
\]

and our final answer is

\[
y(x) = \frac{\pi}{2} \sin x.
\]

(4) (Variant of 1.4 #69) If an idealized cable with uniform density is hung between two points, it will sag until it forms an *catenary* curve. It can be shown that the differential equation governing the shape \(y(x)\) of a cable with endpoints fixed at equal heights is

\[
a \frac{d^2 y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}
\]

where \(a\) is a constant describing the flexibility of the cable. Making use of the substitution \(v = \frac{dy}{dx}\), we can get a first order equation

\[
a \frac{dv}{dx} = \sqrt{1 + v^2}.
\]

Use this equation to show that

\[
y(x) = a \cosh \left(\frac{x}{a}\right) + C.
\]

**Note:** A particularly important example of a catenary is the Gateway Arch here in St. Louis.

**Solution** The equation for \(v\) is separable, and we get

\[
\frac{dv}{\sqrt{1 + v^2}} = \frac{1}{a} \, dx
\]

\[
\int \frac{dv}{\sqrt{1 + v^2}} = \int \frac{1}{a} \, dx
\]

\[
\ln(v + \sqrt{1 + v^2}) = \frac{x}{a} + C_1
\]

See the comment below about the integral! Now if we add the initial condition that \(v(0) = 0\), we have

\[
\ln 1 = 0 + C \implies C_1 = 0
\]
and we get

\[
\ln(v + \sqrt{1 + v^2}) = \frac{x}{a}
\]

\[
v + \sqrt{1 + v^2} = e^{x/a}
\]

\[
\sqrt{1 + v^2} = e^{x/a} - v
\]

\[
1 + v^2 = e^{2x/a} - 2ve^{x/a} + v^2
\]

\[
1 = e^{2x/a} - 2ve^{x/a}
\]

\[
v = \frac{e^{2x/a} - 1}{2e^{x/a}}
\]

\[
v = \frac{e^{x/a} - e^{-x/a}}{2} = \sinh \frac{x}{a}
\]

as desired. Integrating, we have

\[
y = \int v \, dx = \int \sinh \frac{x}{a} \, dx = \cosh x + C
\]

as claimed.

In order to compute the integral, we could use a trigonometric substitution. If we let \( v = \tan t \), \( dv = \sec^2 t \, dt \) and \( \sqrt{1 + v^2} = \sec t \, dt \), we would get

\[
\int \frac{dv}{\sqrt{1 + v^2}} = \int \sec t \, dt
\]

\[
= \ln |\sec t + \tan t|
\]

Now \( \tan t = v \) and \( \sec t = \sqrt{\tan^2 t + 1} = \sqrt{v^2 + 1} \), so we get the claimed integral (note that \( v + \sqrt{v^2 + 1} \) is always positive!).