Instructions: Write your solutions to the following problems and submit them on Crowdmark by the deadline. You are encouraged to work in groups or consult with each other on the problems, but the work submitted must be your own and must be written up by you.

(1) Find a linear homogeneous constant-coefficient equation with the general solution
\[ y(x) = (A + Bx) \cos(2x) + (D + Ex) \sin(2x) + F e^x. \]

Solution. The characteristic polynomial has a complex root of \(2i\), with multiplicity 2 (since we have terms up to \(x\)). The conjugate of this is \(-2i\). We also have a root of 1, so the polynomial is
\[ (r - 2i)^2(r + 2i)^2(r - 1) = (r^2 + 4)^2(r - 1). \]
Expanding this, it is
\[ (r^4 + 8r^2 + 16)(r - 1) = r^5 - r^4 + 8r^3 - 8r^2 + 16r - 16 \]
and the corresponding equation is
\[ y^{(5)} - y^{(4)} + 8y^{(3)} - 8y'' + 16y' - 16y = 0. \]

(2) (a) (3.3 # 26) Solve the initial value problem
\[ y^{(3)} + 10y'' + 25y' = 0; \quad y(0) = 3, y'(0) = 4, y''(0) = 5. \]
(b) (3.3 #32) Find the general solution to
\[ y^{(4)} + y^{(3)} - 3y'' - 5y' - 2y = 0. \]

Solution.
(a) The characteristic polynomial here is
\[ r^3 + 10r^2 + 25r = r(r + 5)^2 \]
which has a root \(r = 0\) and a repeated root at \(r = -5\). The corresponding general solution is
\[ y = A + Be^{-5x} + Cxe^{-5x}. \]
Computing the derivatives, we have
\[
\begin{align*}
y &= A + Be^{-5x} + Cxe^{-5x} \\
y' &= -5Be^{-5x} + Ce^{-5x} - 5Cxe^{-5x} \\
y'' &= 25Be^{-5x} - 10Ce^{-5x} + 25Cxe^{-5x}.
\end{align*}
\]
Setting $x = 0$, we need to solve the following system of equations:

\[
1A + 1B + 0C = 3 \\
0A - 5B + C = 4 \\
0A + 25B - 10C = 5
\]

Substituting $C = 4 + 5B$ into the third equation gives

\[
25B - 40 - 50B = 5 \implies -25B = 45 \implies B = -9/5
\]

Therefore $C = -5$ and $A = 3 - B = 24/5$. Our solution is

\[
y(t) = \frac{24}{5} - \frac{9}{5}e^{-5x} - 5xe^{-5x}.
\]

(b) The characteristic polynomial is

\[
r^4 + r^3 - 3r^2 - 5r - 2
\]

So we have to factor this polynomial. By the rational roots theorem, the possible rational roots are $\pm 1$ and $\pm 2$; we find by trial and error at $r = 2$ is a root. Dividing this polynomial by $r - 2$ (using either synthetic or long division), we find that

\[
r^4 + r^3 - 3r^2 - 5r - 2 = (r - 2)(r^3 + 3r^2 + 3r + 1).
\]

This latter polynomial is just $(r + 1)^3$. The original polynomial therefore has a single root at 2 and a triple root at $-1$; the solution is

\[
y(t) = Ae^{2x} + (B + Cx + Dx^2)e^{-x}.
\]

(3) (3.4 # 4) A body with mass 0.25 kilograms is attached to the end of a spring that is stretched 0.25 m by a force of 9 N. At time $t = 0$, the body is pulled 1 meter to the right, stretching the spring, and set in motion with an initial velocity of 5 m/s to the right.

(a) Find the position $x(t)$.

(b) Write the solution in the form $x(t) = A \cos(\omega_0 t - \delta)$ and find the amplitude of the motion.

**Solution.**

(a) The spring constant is 36 N / m, found by dividing 9 by 0.25. So our model is

\[
0.25x'' + 36x = 0 \implies x'' + 144x = 0
\]

The characteristic polynomial $r^2 + 144$ has roots at $\pm 12i$, and so

\[x(t) = C_1 \cos(12t) + C_2 \sin(12t).
\]

We have the initial conditions $x(0) = 1$ and $x'(0) = -5$, leading to

\[x(t) = \cos(12t) + \frac{5}{12} \sin(12t).
\]

(b) We need to write this in the form $x(t) = A \cos(12t - \delta)$. We have

\[
1 \cos(12t) + \frac{5}{12} \sin(12t) = x(t)
\]

\[A \cos(\delta) \cos(12t) + A \sin(\delta) \sin(12t) = x(t)
\]

Therefore $A^2 = (A \cos \delta)^2 + (A \sin \delta)^2 = 1 + (5/12)^2 = 169/144 \implies A = 13/12$. We also have

\[
\tan \delta = \frac{A \sin \delta}{A \cos \delta} = \frac{5/12}{1} = \frac{5}{12}
\]

An angle with positive sine and cosine is in the first quadrant, so $\delta = \arctan(5/12)$. Thus, our solution is

\[x(t) = \frac{13}{12} \cos \left(12t - \arctan \left(\frac{5}{12}\right)\right).
\]
Consider a mass-spring system with \( m = 1 \) and \( k = 1 \). The mass starts at the equilibrium \( (x(0) = 0) \) and has initial velocity \( x'(0) = 1 \).

(a) What is the maximum displacement from the equilibrium?
(b) Suppose we change the damping from \( c = 0 \) to \( c = 2 \). What is the new maximum displacement?

**Solution.**

(a) The differential equation is \( x'' + x = 0 \) \( \Rightarrow \) \( x'' + x = 0 \) and the solution is

\[
x = C_1 \cos t + C_2 \sin t
\]

Since \( x(0) = 0 \), the cosine term vanishes. Since \( x' = C_2 \cos t \), we get \( C_2 = 1 \). Thus,

\[
x(t) = \sin t
\]

which has amplitude 1. This is the maximum displacement.

(b) The new differential equation is \( x'' + 2x' + x = 0 \). The characteristic polynomial is

\[
r^2 + 2r + 1 = 0
\]

which has a repeated root \( r = 1 \). The general solution is

\[
x = C_1 e^{-t} + C_2 te^{-t}.
\]

Since \( x(0) = 0 \), we get \( C_1 = 0 \). Therefore

\[
x' = C_2 e^{-t} - C_2 te^{-t}
\]

and the value \( x'(0) = 1 \) leads us to \( C_2 = 1 \). Therefore, \( x(t) = te^{-t} \). We can check that this is maximized when \( t = 1 \), and the displacement is \( 1/e \approx 0.37 \). This is quite a bit smaller than the undamped case!