Wednesday 9/5 — Slope fields, existence – uniqueness.

We saw \( y' = x - y \): 

\[
\begin{align*}
\text{Explore plots like this a GeoGebra!!}
\end{align*}
\]

Note along \( y = x + 1 \), \( y' = 1 \). So we’d stay on this line.

We get a steady growth — hypothesize our solution is \( y(x) = x - 1 + \text{error} \). In fact, solution is \( x - 1 + C e^{-x} \).

Existence – Uniqueness.

\[
\begin{align*}
y' &= \frac{1}{x}, \quad y(0) = 0 \text{ has no solution.}
\end{align*}
\]

\[
\begin{align*}
y' &= 2\sqrt{y}, \quad y(0) = 0 \text{ has two solutions}
\end{align*}
\]

\[
\begin{align*}
y_1(x) &= x^2 \\
y_2(x) &= 0.
\end{align*}
\]

Both situations are BAD.

Theorem: \( y' = f(x, y) \), \( y(a) = b \) has a unique solution on an interval around \( a \) if \( f \) and \( \partial_y f \) are both continuous on a rectangle containing \((a, b)\).

\[
\begin{align*}
\text{Ex.} \quad y' &= 2\sqrt{y} \quad \rightarrow \quad f(x, y) = 2\sqrt{y} \text{ continuous for } y \geq 0 \\
&\text{but } \frac{\partial y}{\partial y} = \frac{1}{2\sqrt{y}} \text{ is not continuous at } y = 0.
\end{align*}
\]

\[
\begin{align*}
\text{Ex.} \quad y' &= x - y \quad \rightarrow \quad f(x, y) = x - y \text{ and } \partial_y f = -1 \text{ are fine everywhere. So get a solution...}
\end{align*}
\]