Homework 5

Math 308

Due: 1 March

Guidelines:

- You are strongly encouraged to work together to understand the problems, but what you turn in must be your own work.
- Your submission must be clearly written and stapled. Homework will only be accepted up to the beginning of lecture, or you can drop it off at my office before class.

(1) (6.2.2)

(a) Show that if f and g are functions with second derivatives, then $u = f(x - vt) + g(x + vt)$ is a solution of the wave equation; this is called the d'Alembert solution and is physically interpreted as a superposition of two waves moving in opposite directions.

(b) Show that $u(r,t) = \frac{1}{r}f(r - vt) + \frac{1}{r}g(r + vt)$ is a solution of the wave equation in spherical coordinates. (See, e.g. p. 294 for the gradient in spherical coordinates). Again, this can be interpreted in terms of waves moving in and out from the origin.

(2) (See 6.3.2) A bar of length 10 centimeters is initially at 100°, and starting at time $t = 0$ the ends are held at 0°. Find the temperature distribution at time t. Take the thermal diffusivity quantity $\alpha = 1$. For any time t, determine where the maximum temperature in the bar is. Roughly how long does it take for the temperature to drop to 1°? What about 0.1°?

(3) (See 6.3.8) A bar of length 2 is initially at temperature 0°. From $t = 0$ on, the $x = 0$ end is held at 0° and the $x = 2$ end is held at 100°. Take the thermal diffusivity quantity $\alpha = 1$. Find the time dependent temperature distribution $u(x,t)$. Then compute $\lim_{t\to\infty} u(x,t)$ for any x; explain what this means physically.

(4) (6.2.16) Suppose we have a closed region D and a harmonic function u which takes certain values on the boundary of the region. That is, we have a solution to the PDE

$$\begin{cases} \nabla^2 u &= 0 \quad \text{in } D \\ \quad u &= f \quad \text{on } \partial D \end{cases}$$

Show that there is only one such function (that is, the solution with given boundary data is *unique*). To do this, suppose that u_1 and u_2 are two solutions; then $U := u_1 - u_2$ satisfies Laplace's equation with zero boundary data. Use Green's first identity (from a previous homework) with the function U to show that $\nabla U \equiv 0$. Then explain why this is enough.