Homework 8

Math 308

Due: 1 April

Guidelines:

- You are strongly encouraged to work together to understand the problems, but what you turn in must be your own work.
- Your submission must be clearly written and stapled. Homework will only be accepted up to the beginning of lecture, or you can drop it off at my office before class.

(1) Verify that $\nabla^2(1/r) = 0$ for $r \neq 0$. (Note that there is an easy way and a hard way for this problem.)

(2) (13.5.10) A cube is initially held at constant temperature T_0, and all its faces are held at $0°$. The thermal diffusivity constant is again $\alpha = 1$. Determine the time-dependent temperature distribution in the cube; how long does it take for the maximum temperature in the cube to fall to $\frac{1}{2}T_0$? You will need to expand the solution in terms of a triple Fourier series similar to the wave-on-a-rectangle example in class.

(3) This exercise introduces a fundamentally important family of orthogonal functions, called the *Legendre polynomials*. The polynomials $\{P_n : n = 0, 1, 2, 3, ...\}$ are each degree n polynomials that satisfy the equation

$$\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}P_n\right] + n(n+1)P_n = 0$$

and are closely connected with spherical harmonics. They satisfy the following properties:

- $P_0(x) = 1$ for all x.
- The polynomials are orthogonal on $[-1, 1]$, meaning that $\int_{-1}^{1} P_n P_m \, dx = 0$ if $n \neq m$.
- They satisfy the boundary conditions $P_n(1) = 1$ and $P_n(-1) = (-1)^n$.

Use this information to compute the Legendre polynomials of degree at most 4.

(4) Continuing the previous problem, the Legendre polynomial also satisfy the normalization

$$\int_{-1}^{1} P_m P_n \, dx = \frac{2}{2n+1}\delta_{mn}$$

where δ_{mn} is the Kronecker delta, which is zero for $m \neq n$ and 1 if $m = n$.

It is possible to expand a continuous function as a *Fourier-Legendre series* of the form

$$f(x) = \sum_{n=0}^{\infty} \alpha_n P_n(x).$$

Write a formula for α_n in terms of integrals, and then explicitly compute the first five terms of the Fourier-Legendre series for the square wave ($S(x) = 0$ for $-1 < x < 0$ and $S(x) = 1$ for $0 < x < 1$).