\[ \pi = 3.141592654, \quad \ln(1) = 0, \quad \ln(e) = 1, \quad \ln(x y) = \ln(x) + \ln(y), \quad \ln(x^p) = p \ln(x) \]
\[ \sin \left( \frac{\pi}{6} \right) = \cos \left( \frac{\pi}{3} \right) = \frac{1}{2}, \quad \sin \left( \frac{\pi}{3} \right) = \cos \left( \frac{\pi}{6} \right) = \frac{\sqrt{3}}{2}, \quad \sin \left( \frac{\pi}{4} \right) = \cos \left( \frac{\pi}{4} \right) = \frac{\sqrt{2}}{2}, \]
\[ \sin(0) = \sin(\pi) = \cos \left( \frac{\pi}{2} \right) = 0, \quad \sin \left( \frac{\pi}{2} \right) = \cos(0) = \tan \left( \frac{\pi}{4} \right) = 1, \quad \sin \left( \frac{3\pi}{2} \right) = \cos(\pi) = -1 \]

\[ \sin(2x) = 2 \sin(x) \cos(x), \quad 2 \cos(x)^2 = 1 + \cos(2x), \quad 2 \sin(x)^2 = 1 - \cos(2x), \]
\[ \int \frac{1}{x} \, dx = \ln(|x|) + C, \quad \int \ln(x) \, dx = x \ln(x) - x + C, \quad \int a^x \, dx = \frac{a^x}{\ln(a)} + C, \]
\[ \int u \, dv = uv - \int v \, du \]
\[ \int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \arctan \left( \frac{x}{a} \right) + C, \quad \int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \arcsin \left( \frac{x}{a} \right) + C, \]
\[ \int \frac{1}{|x| \sqrt{x^2 - a^2}} \, dx = \frac{1}{a} \text{arcsec} \left( \frac{x}{a} \right) + C, \quad 1 - \sin(x)^2 = \cos(x)^2, \quad 1 + \tan(x)^2 = \sec(x)^2 \]
\[ \int \sec(x)^2 \, dx = \tan(x) + C, \quad \int \csc(x)^2 \, dx = -\cot(x) + C, \]
\[ \int 2 \sec(x)^3 \, dx = \sec(x) \tan(x) + \ln(|\sec(x) + \tan(x)|) + C \]
\[ \int \sec(x) \tan(x) \, dx = \sec(x) + C, \quad \int \csc(x) \cot(x) \, dx = -\csc(x) + C \]
\[ \int \tan(x) \, dx = \ln(|\sec(x)|) + C, \quad \int \sec(x) \, dx = \ln(|\sec(x) + \tan(x)|) + C \]
\[ \int \cos^n(x) \, dx = \frac{1}{n} \sin(x) \cos^{(n-1)}(x) + \frac{n-1}{n} \int \cos^{(n-2)}(x) \, dx \]

Integrating factor for \[ \frac{dy}{dx} + p(x) y = q(x) : \quad e^{\int p(x) \, dx} \quad \text{where} \quad P(x) = \int p(x) \, dx \]

1. Let \( R \) be the region that is bounded above by \( y = 2\sqrt{x} \) and below by \( y = x \).

   The area of \( R \) is \( \frac{8}{3} \) and the mass density of \( R \) is the constant \( \delta \) at every point.

   What is the x-coordinate of the center of mass of \( R \)?
Solution: d

Mass

\[
A := \int_0^4 (2\sqrt{x} - x) \, dx
\]

\[
A := \frac{8}{3}
\]

Moment about y-axis (denoted here by My)

\[
My := \int_0^4 x(2\sqrt{x} - x) \delta \, dx
\]

\[
My := \frac{64}{15} \delta
\]

x-coordinate of center of mass (denoted here by x_bar)

\[
x_{\text{bar}} := \frac{My}{M}
\]
The x-coordinate of the center of mass is the moment about the y-axis divided by the area.

\[ x_{\text{bar}} = \frac{8}{5} \]

The center of mass lies on the vertical line drawn:

2. As in the preceding problem, let \( R \) be the region that is bounded above by \( y = 2 \sqrt{x} \) and below by \( y = x \). The area of \( R \) is \( \frac{8}{3} \) and the mass density of \( R \) is the constant \( \delta \) at every point. What is the y-coordinate of the center of mass of \( R \)?

a) \( \frac{8}{5} \)

b) \( \frac{5}{3} \)

c) \( \frac{26}{15} \)

d) \( \frac{9}{5} \)

e) \( \frac{28}{15} \)

f) \( \frac{29}{15} \)

g) \( 2 \)

h) \( \frac{31}{15} \)

i) \( \frac{32}{15} \)

j) \( \frac{11}{5} \)

Solution: \( g \)

Moment about the x-axis (denoted here by \( M_x \))
\[ M_x := \int_0^4 (4x - x^2) \delta \, dx \]

\[ M_x := \frac{32 \delta}{3} \]

y-coordinate of center of mass (denoted here by \( y_{\text{bar}} \))

\[ y_{\text{bar}} := \frac{M_x}{2M} \]

The y-center of mass lies on the horizontal line drawn. The center of mass is in the cross-hairs.
3. A function $y(x)$ that has positive values for $x > 0$ satisfies the initial value problem

$$2x^2 \frac{dy}{dx} = \frac{x^2 - 2}{y + 1}, \quad y(1) = 3.$$ 

What is $y(2)$?

a) 3  b) 4  c) 5  d) 6  e) 7
f) 8  g) 9  h) 10  i) 11  j) 12

**Solution: a**

After separating variables, the integrated equation is

```plaintext
> eqn1 := Int(2*(y + 1), y) = Int((x^2-2)/x^2, x);

\[
\int 2y + 2 \, dy = \int \frac{x^2 - 2}{x^2} \, dx
\]

> eqn2 := lhs(eqn1) = expand(rhs(eqn1));

\[
\int 2y + 2 \, dy = \int 1 - \frac{2}{x^2} \, dx
\]

> eqn3 := value(lhs(eqn2)) = value(rhs(eqn2)) + C;

\[
y^2 + 2y = x + \frac{2}{x} + C
\]

> eqn4 := lhs(eqn3) + 1 = rhs(eqn3);

\[
y^2 + 2y + 1 = x + \frac{2}{x} + C
\]

> eqn5 := factor(lhs(eqn4)) = rhs(eqn4);

\[
(y + 1)^2 = x + \frac{2}{x} + C
\]

> initialCondition := subs({x=1,y=3}, eqn5);

\[
initialCondition := 16 = 3 + C
\]

> eqn_for_C := C = solve(initialCondition, C);

\[
eqn_for_C := C = 13
\]

> eqn6 := subs(eqn_for_C,eqn5);

\[
eqn6 := (y + 1)^2 = x + \frac{2}{x} + 13
\]

> eqn7 := map(simplify, subs(x=2,eqn6));

\[
eqn7 := (y + 1)^2 = 16
\]
```
The function \( y(x) \) we seek takes positive values, we have been told, and, in any event, \(-5\) is not among the answer choices.

We now use Maple's differential equation solver for confirmation of our work.
4. A mixing tank initially held 120 liters of water in which \( q \) kilograms of a solute were dissolved. In order to reduce the strength of the solution, valves were opened so that there was an inflow of water at the rate of 15 liters per minute and an outflow of the thoroughly mixed solution at the rate of 15 liters per minute. The valves were closed as soon as the mass of the solute in the tank had been reduced to \( q e^{-3} \) kilograms. For how many minutes were the valves open?

a) 12   b) 15   c) 16   d) 18   e) 20
f) 24   g) 27   h) 28   i) 30   j) 32

Solution: f

Let \( y(t) \) be the mass of the solute in the tank at time \( t \), (in minutes).

\[
\text{ode} := \text{diff}(y(t),t) = 0 - \left(\frac{y(t)}{120}\right) \times 15;
\]

\[
\text{ode} := \frac{dy}{dt} = -\frac{1}{8} y(t)
\]
linear_ode := map( z -> z + y(t)/8, ode);

\[ \text{linear_ode} := \left( \frac{d}{dt}y(t) \right) + \frac{1}{8}y(t) = 0 \]

linear_ode_times_integrating_factor := map( z -> expand(exp(int(1/8, t))*z), linear_ode);

\[ \text{linear_ode_times_integrating_factor} := e^{\left( \frac{t}{8} \right)} \left( \frac{d}{dt}y(t) \right) + \frac{1}{8}e^{\left( \frac{t}{8} \right)} y(t) = 0 \]

soln_eqn1 := Diff( exp(t/8)*y(t), t) = 0;

\[ \text{soln_eqn1} := \frac{d}{dt} \left( e^{\left( \frac{t}{8} \right)} y(t) \right) = 0 \]

soln_eqn2 := exp(t/8)*y(t) = C;

\[ \text{soln_eqn2} := e^{\left( \frac{t}{8} \right)} y(t) = C \]

eqn_for_C := C = subs(y(0)=q, simplify(subs(t=0, lhs(soln_eqn2))));

\[ \text{eqn_for_C} := C = q \]

soln_eqn3 := subs(eqn_for_C, soln_eqn2);

\[ \text{soln_eqn3} := e^{\left( \frac{t}{8} \right)} y(t) = q \]

soln_eqn4 := y(t) = solve(soln_eqn3, y(t));

\[ \text{soln_eqn4} := y(t) = \frac{q}{e^{\left( \frac{t}{8} \right)}} \]

answer := solve(subs(y(t) = q*exp(-3), soln_eqn4), t);

\[ \text{answer} := 24 \]

Confirmation using Maple's differential equation solver.

eqn := dsolve( \{diff(y(t),t) = 0 - 15*y(t)/120, y(0) = q\}, y(t));

\[ \text{eqn} := y(t) = q e^{\left( \frac{-t}{8} \right)} \]

eqn := y(t) = q*exp(-1/8*t);

solve(q*exp(-3) = rhs(eqn), t);

\[ 24 \]

5. A mixing tank initially held 120 liters of water in which 15 kilograms of salt were dissolved. Valves were opened so that there was an inflow
of a salt water solution at the rate of 20 liters per minute and an outflow of the thoroughly mixed solution at the same rate of 20 liters per minute. The concentration of the solution entering the mixing tank was 1 kilogram of salt per 5 liters of water. For how many minutes had the valves been open at the moment the mass of salt in the mixing tank was 21 kilograms?

a) 2 \ln(2)  

b) 3 \ln(2)  

c) 4 \ln(2)  

d) 5 \ln(2)  

e) 6 \ln(2)  

f) 2 \ln(3)  

g) 3 \ln(3)  

h) 4 \ln(3)  

i) 5 \ln(3)  

j) 6 \ln(3)  

Solution:  

\[ t = 6 \ln(3) \]

6. A large (250 liter capacity), partially-filled mixing tank initially held 120 liters of water in which 16 kilograms of a solute were dissolved. Two valves were simultaneously opened. The opening of one valve resulted in the flow of a solution into the mixing tank at the rate of 2 liters per minute. The solution entering the mixing tank contained the same solute as was in the tank. The concentration of the solution entering the mixing tank was 1 kilogram per 2 liters. The opening of the other valve allowed the thoroughly mixed solution to flow out of the tank at the rate of 1 liter per minute. What was the mass in kilograms of the solute in the mixing tank 2 hours after the valves were opened?

a) 82  

b) 84  

c) 86  

d) 88  

e) 90  

f) 92  

g) 94  

h) 96  

i) 98  

j) 100  

Solution:  

\[ t = 6 \ln(3) \]
> IVP := {ode, ic};

\[
IVP := \{ y(0) = 16, \frac{dy}{dt} = 1 - \frac{y(t)}{120 + t} \}
\]

> dsolve(IVP,y(t));

\[
y(t) = \frac{120 t + \frac{t^2}{2} + 1920}{120 + t}
\]

> subs(t = 120, rhs(%));

98

7. A population \( P(t) \) grew according to the differential equation

\[
\frac{dP}{dt} = kP
\]

for some positive constant \( k \). The initial population was 800. Sixteen years later, the population was 1600. What was the instantaneous rate of change of the population (in hundreds of population units per year) thirty-two years after the initial count?

a) \( \frac{1}{2} \ln(2) \)

b) \( \ln(2) \)

c) \( 2 \ln(2) \)

d) \( 4 \ln(2) \)

e) \( 8 \ln(2) \)

f) \( 16 \ln(2) \)

g) \( 32 \ln(2) \)

h) \( 64 \ln(2) \)

i) \( 128 \ln(2) \)

j) \( 256 \ln(2) \)

Solution: c

> T := 16;

# T is the doubling time

\( T := 16 \)

> eqn1 := k = solve(k*T = ln(2), k);

# Use the doubling time to solve for the growth constant

\[
eqn1 := k = \frac{1}{16} \ln(2)
\]

> eqn2 := P(t) = 800*exp(k*t);

# The equation for the population size (Alternatively, \( P(t) = 800*2^{(t/16)} \))

\[
eqn2 := P(t) = 800 e^{(kt)}
\]

> eqn3 := D(P)(t) = k*P(t);

# equation for the instantaneous rate of change (measured in population units per year)
\[
\text{eqn3} := \frac{dP}{dt} = kP(t)
\]

\[
\text{eqn4} := \frac{dP}{dt} = 800 k e^{(k t)}
\]

\[
\text{eqn5} := \frac{dP}{dt} = 50 \ln(2) e^{(1/16 \ln(2) \cdot t)}
\]

\[
\text{eqn6} := \frac{dP}{dt} = 200 \ln(2)
\]

8. A population \(P(t)\) with initial population 300 grew according to the differential equation

\[
\frac{dP}{dt} = 7 P(t) (2100 - P(t)).
\]

What is

\[
\lim_{t \to \infty} P(t) + \lim_{t \to \infty} \frac{dP}{dt}?
\]

a) 600  b) 700  c) 900  d) 1000  e) 1400  

f) 1600  g) 2100  h) 2400  i) 2700  j) 2800

Solution: g

The differential equation models logistic growth. (In the presentation found in
Stewart's text, the 2100 would be factored from the parentheses and combined with the 7 to form the equivalent equation \( \frac{dP}{dt} = 14700 \left( 1 - \frac{P}{2100} \right) \).

The solution has the properties that

\[
\lim_{t \to \infty} P(t) = 2100 \quad \text{and} \quad \lim_{t \to \infty} D(P)(t) = 0.
\]

9. If \( y(x) \) is the solution to the initial value problem

\[
\frac{dy}{dx} + \frac{3y}{x} = \frac{38}{x^2}, \quad y(2) = 5
\]

then what is \( y(6) \)?

a) 1  b) 2  c) 3  d) 4  e) 5  f) 6  g) 7  h) 8  i) 9  j) 10

Solution: c

```maple
> ode := diff(y(x),x) + 3/x*y(x) = 38/x^2;
  ic := y(2) = 5;
  IVP := {ode,ic};
  ode := \left( \frac{d}{dx} \frac{y(x)}{x} \right) + \frac{3 y(x)}{x} = \frac{38}{x^2}
  ic := y(2) = 5
  IVP := \{ \left( \frac{d}{dx} \frac{y(x)}{x} \right) + \frac{3 y(x)}{x} = \frac{38}{x^2}; y(2) = 5 \}

> dsolve(IVP,y(x));
  y(x) = \frac{19 x^2 - 36}{x^3}

> subs(x=6, (19*x^2-36)/x^3);
  \frac{3}{x}

> plot((19*x^2-36)/x^3, x = 2.5..6, thickness=2, color=NAVY, view=[2..6,2..5.5]);
```
10. If $y(x)$ is the solution to the initial value problem

$$\frac{dy}{dx} = 2y(x) + e^{3x}, \quad y(0) = 0$$

then what is $y(1)$?

a) $e$  

b) $e(e-1)$

c) $e(e+1)$

d) $e^2$

e) $e^2(e-1)$

f) $e^2(e+1)$

g) $e^3$

h) $e^3(e-1)$

i) $e^3(e+1)$

j) $3e$

Solution: $e$
\[ IVP : = \{ \frac{d}{dx} y(x) = 2 y(x) + e^{(3x)} , y(0) = 0 \} \]

```maple
> dsolve(IVP, y(x));
\]
> subs(x=1, %);
\]
> curve := plot([\((\exp(x)-1)\)*\exp(2*x), (\exp(1)-1)*\exp(2)\], x=0 .. 1.1, color = [NAVY, GREY], thickness=[2,1]):
vert := plot([1,y,y=0..subs(x=1.1,(\exp(x)-1)*\exp(2*x))], color = GREY):
plots[display](curve,vert);
```

11. For nonzero constants \( a \) and \( b \), and for the sequence defined by

\[ x_n = \sqrt{\frac{9 n^5 50 + 200 b^2}{4 n^5 50 + 200 a^2}}, \]

calculate \( \lim_{n \to \infty} x_n \).

a) \( \frac{3}{2} \)  b) \( \frac{9}{4} \)  c) \( \frac{b}{a} \)  d) \( \frac{b^2}{a^2} \)  e) \( \sqrt{\frac{9 + b^2}{4 + a^2}} \)
\[ f) \quad \frac{9 + b^2}{4 + a^2} \quad g) \quad \frac{3 + b}{2 + a} \quad h) \quad \frac{3b}{2a} \quad i) \quad 0 \quad j) \quad \text{Does not exist} \]

**Solution: a**

Divide each term in the fraction by \( n^{50} \):

\[
x_n = \sqrt[50]{\frac{9 + \frac{c}{n^{50}}}{4 + \frac{d}{n^{50}}}} \quad \text{where} \quad c = 100^{200} b^2 \quad \text{and} \quad d = 200^{100} a^2 \quad \text{are constants.}
\]

It follows that

\[
\lim_{n \to \infty} x_n = \sqrt[50]{\frac{9 + 0}{4 + 0}} = \frac{3}{2}.
\]

**12.** Let \( r \) be a constant. Calculate

\[
\lim_{n \to \infty} \left( 1 - \frac{r}{n} \right)^n.
\]

a) \( 1 \)  \quad b) \( r \)  \quad c) \( 1 - r \)  \quad d) \( 1 - \frac{1}{r} \)  \quad e) \( \ln(r) \)

f) \( \frac{1}{\ln(r)} \)  \quad g) \( e^r \)  \quad h) \( \frac{1}{e^r} \)  \quad i) \( 0 \)  \quad j) \( \text{Does not exist} \)

**Solution: h**

An important limit formula is

\[
\lim_{n \to \infty} \left( 1 + \frac{a}{n} \right)^n = e^a \quad \text{for any constant} \quad a.
\]

To obtain this formula,

let \( y_n = \ln \left( \left( 1 + \frac{a}{n} \right)^n \right) = n \ln \left( 1 + \frac{a}{n} \right) = \frac{\ln \left( 1 + \frac{a}{n} \right)}{\frac{1}{n}} = \frac{\ln(1 + au)}{u} \quad \text{where} \quad u = \frac{1}{n} \quad \text{tends to} \quad 0
\]

from the right.

By L'Hopital's Rule,
\[
\lim_\limits_{n \to \infty} y_n = \lim_\limits_{u \to 0^+} \frac{\partial}{\partial u} \ln(1 + a u) = \lim_\limits_{u \to 0^+} \frac{a}{1 + a u} = a
\]

It follows that
\[
\lim_\limits_{n \to \infty} \left(1 + \frac{a}{n}\right)^n = \lim_\limits_{n \to \infty} e^{y_n} = e^a.
\]

Now set \( a = -r \) to answer this problem.

\[13. \text{ For positive constants } a \text{ and } b, \text{ and for the sequence } \{x_n\} \text{ defined by}
\]
\[
x_n = \left(1 + \frac{1}{n}\right)^{a + e^{b + n}} \quad \text{ (the } n'\text{th root of } a + e^{b + n}),
\]
calculate \( \lim_\limits{n \to \infty} x_n \).

a) \( a \) \quad b) \( e \) \quad c) \( a + e \) \quad d) \( e^b \) \quad e) \( a + e^b \)
f) \( a e \) \quad g) \( a e^b \) \quad h) \( 1 \) \quad i) \( 0 \) \quad j) \text{ Does not exist}

\textbf{Solution: } b

Intuitively, for large values of \( n \) we expect the additive constants \( a \) and \( b \) to have the effect of 0,
\[
\left(\frac{1}{n}\right)
\]
leading us to expect \( x_n \) to be approximately \( (e^n)^{\frac{1}{n}} \), or \( e \).

Let \( y_n = \ln(x_n) = \frac{\ln(a + e^{b + n})}{n} \).

By L’Hopital’s Rule we have
\[
\lim_\limits{n \to \infty} y_n = \lim_\limits{n \to \infty} \frac{\frac{d}{dn} \ln(a + e^{b + n})}{\frac{d}{dn} n} = \lim_\limits{n \to \infty} \frac{e^{b + n}}{a + e^{b + n}} = \lim_\limits{n \to \infty} \frac{1}{e^{b + n} + 1} = \frac{1}{0 + 1} = 1.
\]

It follows that
\[
\lim_\limits{n \to \infty} x_n = \lim_\limits{n \to \infty} e^{y_n} = e = e.
\]
Maple answers this question with a one-line command:

```maple
> Limit( (a+exp(b + n))^(1/n), n = infinity) = limit( (a+exp(b + n))^(1/n), n = infinity);

\[ \lim_{n \to \infty} (a + e^{(b+n)})^{1/n} = e \]
```

14. Let \( \{ s_N \} \) denote the sequence of partial sums of the series \( \sum_{n=1}^{\infty} a_n \).

If \( s_N = \left( \frac{\ln(N)}{\sqrt{N}} + 192 N^5 \right)^{\frac{1}{3}} \left( \frac{1}{N} \right) + 3 N^5 \), what is the value of \( \sum_{n=1}^{\infty} a_n \)?

a) 1    b) 2    c) 3    d) 4    e) 5
f) 6    g) 7    h) 8    i) 0    j) The series diverges

Solution: d

Divide every term in the fraction by \( n^5 \): 

\[ s_N = \left( \frac{\ln(N)}{\sqrt{N}} + 192 \right)^{\frac{1}{3}} \left( \frac{1}{N} \right) + 3 N^5 \]
Because \( \lim_{N \to \infty} \frac{\ln(N)}{N^p} = 0 \) for any positive value of \( p \), and \( \lim_{N \to \infty} \left( \frac{1}{N} \right) = 1 \), we have

\[
\lim_{N \to \infty} s_N = \left( \frac{0 + 192}{0 + 3} \right) \left( \frac{1}{3} \right) = 64 \left( \frac{1}{3} \right) = 4.
\]

15. Wacko and Dot are calculating the sequence \( \{s_N\} \) of partial sums of the series

\[
\sum_{n=1}^{\infty} \frac{n}{2^n}.
\]

They split up the labor with Wacko calculating \( s_N \) for odd values of \( N \) and Dot calculating \( s_N \) for even \( N \). Almost immediately after Wacko announces \( s_9 = \frac{1013}{512} \), Dot correctly follows with what value for \( s_{10} \)?

a) \( \frac{2027}{1024} \)  
   b) \( \frac{507}{256} \)  
   c) \( \frac{2029}{1024} \)  
   d) \( \frac{1015}{512} \)  
   e) \( \frac{2031}{1024} \)  
   f) \( \frac{127}{64} \)  
   g) \( \frac{2033}{1024} \)  
   h) \( \frac{1017}{512} \)  
   i) \( \frac{2035}{1024} \)  
   j) \( \frac{509}{256} \)

Solution: j

\[
> \text{eqn1 := } S[9] = \text{sum}(k/2^k, k=1..9); \\
> \text{eqn1 := } S_9 = \frac{1013}{512}
\]

\[
> \text{eqn2 := } S_{10} = S_9 + \frac{5}{512}
\]
16. Poetry Corner

Big fleas have little fleas,
   Upon their backs to bite 'em.
And little fleas have lesser fleas,
   And so \textit{ad infinitum}.

If the mass of each biter is 0.82 times the mass of its bitee, and if the entire mass of the flea chain is 0.572 grams, then, to three decimal places, what is the mass in grams of the largest flea?

a) 0.058   b) 0.067   c) 0.076   d) 0.085   e) 0.094
f) 0.103   g) 0.112   h) 0.121   i) 0.130   j) 0.138

\textbf{Solution: } f

Let $m$ be the mass of the largest flea. Then the mass of the second largest flea in the biting chain is $0.82m$. The mass of the third largest flea in the biting chain is $0.82^2 m$. And so on.

\begin{verbatim}
> subs(eqn1, eqn2);

S_{10} = \frac{509}{256}
\end{verbatim}

17. To which of the following series does the Test for Divergence lead to a conclusion?

\begin{verbatim}
> eqn1 := 0.572 = sum(m*0.82^n, n=0..infinity);

\text{eqn1} := 0.572 = \sum_{n=0}^{\infty} m \cdot 0.82^n
\end{verbatim}

\begin{verbatim}
> eqn2 := 0.572 = sum(m*0.82^n, n=0..infinity);

\text{eqn2} := 0.572 = 5.555555556 m
\end{verbatim}

\begin{verbatim}
> m = solve(eqn2, m);

m = 0.1029600000
\end{verbatim}
\[
\begin{align*}
(i) \sum_{n=1}^{\infty} \frac{1}{n} & \quad (ii) \sum_{n=1}^{\infty} \frac{1 + \sqrt{n}}{n^2} & \quad (iii) \sum_{n=1}^{\infty} \frac{1}{n} & \quad (iv) \sum_{n=1}^{\infty} \frac{2^n + 1}{2^n + n^3}
\end{align*}
\]

a ) (i) only  b ) (ii) only  c ) (iii) only  d ) (iv) only  e ) (i) & (ii) only
f ) (i) & (iii) only  g ) (i) & (iv) only  h ) (ii) & (iii) only  i ) (ii) & (iv) only  j ) (iii) & (iv) only

**Solution:** j

For \( a_n = \frac{1}{n} \) and \( a_n = \frac{1 + \sqrt{n}}{n^2} \), we have \( \lim_{n \to \infty} a_n = 0 \).

The Test for Divergence therefore does not apply to Series ( i ) and ( ii ).

For \( a_n = \frac{1}{n} \) and \( a_n = \frac{2^n + 1}{2^n + n^3} \), we have \( \lim_{n \to \infty} a_n = 1 \).

The Test for Divergence therefore does apply to Series ( iii ) and ( iv )
(with a conclusion of "Diverges").

18. The Integral Test can be used to show that the series

\[
\sum_{n=1}^{\infty} \frac{2n}{(n^2 + 4)^2}
\]

(Note: \( n^2 + 4 \) is squared.)

converges. Let \( s \) be the value of this series. Adding the first 6 terms of the
series results in the sum 0.2173935650 (to ten decimal places). The Integral Test
provides a number \( \beta \) such that \( s \leq 0.2173935650 + \beta \). What is \( \beta \)?
(The question does not ask for some number \( \beta \) that makes the inequality
correct. It asks for the specific number that arises from the Integral Test.
If the desired value for \( \beta \) is replaced with a larger number, then the inequality
remains true. But the replacement value does not correctly answer the question.)

a ) \( \frac{1}{25} \)  b ) \( \frac{1}{30} \)  c ) \( \frac{1}{35} \)  d ) \( \frac{1}{40} \)  e ) \( \frac{1}{45} \)
f ) \( \frac{1}{50} \)  g ) \( \frac{1}{55} \)  h ) \( \frac{1}{60} \)  i ) \( \frac{1}{65} \)  j ) \( \frac{1}{70} \)
Solution: d

The Integral Test gives the value
\[
\beta = \int_{6}^{\infty} \frac{2x}{(x^2 + 4)^2} dx.
\]

> evalf(1/40);
0.02500000000

> evalf(sum(2*n/((n^2+4)^2),n=1..6),20);
0.21739356500080912411

> Int(2*x/((x^2+4)^2),x=6..infinity) = int(2*x/((x^2+4)^2),x=6..infinity);

To ten decimal places, we have
\[
\sum_{n=1}^{\infty} \frac{2n}{(n^2 + 4)^2} = 0.2389127750, \text{ whereas } 0.2173935650 + \frac{1}{40} = 0.2423935650.
\]

19. For which of the following series does the Limit Comparison Test, applied with a p-series for the comparison series, lead to a conclusion of convergence?

(a) (i) only     (b) (ii) only     (c) (iii) only     (d) (iv) only     (e) (i) & (ii) only
(f) (i) & (iii) only   (g) (i) & (iv) only   (h) (ii) & (iii) only   (i) (ii) & (iv) only   (j) (iii) & (iv) only

Solution: b

Series (i) is divergent: compare with \[\sum_{n=1}^{\infty} \frac{1}{2n}.\] No conclusion of convergence is possible.
Series (ii) is proven to be convergent by using the Limit Comparison Test with the p-series
\[ \sum_{n=1}^{\infty} \frac{1}{\left(\frac{3}{2}\right)^n} : \]
for \( a_n = \frac{1 + \sqrt{n}}{n^2} \) and \( b_n = \frac{1}{\left(\frac{3}{2}\right)^n} \), we have
\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{\left(\frac{3}{2}\right)^n + n^2}{n^2} = \lim_{n \to \infty} \frac{n^2 + n^2}{n^2} = \lim_{n \to \infty} \frac{1}{\sqrt{n}} + 1 = 1.
\]
The Limit Comparison Test does work with Series (iii), but not using a p-series for comparison.
For \( a_n = \frac{2^n + n}{4^n + n^3} \) and \( b_n = \frac{1}{n^p} \), we have
\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{(2^n + n)n^p}{4^n + n^3} = \lim_{n \to \infty} \frac{2^n n^p + n^{(p+1)}}{4^n + n^3} = \lim_{n \to \infty} \frac{n^p + n^{(p+1)}}{1 + n^3} = 0 + 0 = 0.
\]
This limit (along with \( \infty \)) is not included in the decisive case of the Limit Comparison Test.
Series (iv) is an alternating series. It has infinitely many negative terms and the Limit Comparison Test therefore does not apply: it cannot even be applied to any tail.

20. Let \( a_n = \frac{1}{n} \) and \( b_n = \frac{2}{n^2} \). Consider the following four alternating series:
(i) \( a_1 - b_2 + a_3 - b_4 + a_5 - b_6 + \ldots \)
(ii) \( a_1 b_1 - a_2 b_2 + a_3 b_3 - a_4 b_4 + \ldots \)
(iii) \( \frac{b_1}{a_1} + \frac{b_3}{a_3} - \frac{b_4}{a_4} + \ldots \)
(iv) \( a_1 - b_2 + \frac{a_3}{3} - b_4 + \frac{a_5}{5} - b_6 + \ldots \)
For which of these series does the Alternating Series Test lead to a conclusion?
To apply the Alternating Series Test to the infinite series $c_k + c_{k+1} + c_{k+2} + \ldots$, two conditions must be satisfied:

(i) $c_{n+1} < c_n$ for all $k \leq n < \infty$ (i.e. the terms decrease)

and

(ii) $\lim_{n \to \infty} c_n = 0$.

```plaintext
> a := n -> 1/n;
b := n -> 2/n^2;
x1 := n -> if is(n,odd) then a(n) else b(n) fi;
x2 := n -> a(n)*b(n);
x3 := n -> b(n)/a(n);
x4 := n -> if is(n,odd) then a(n)/n else b(n) fi;

# x[1], x[2], x[3], x[4] are the four sequences of terms that
generate the four series under consideration

a := n -> \frac{1}{n}
b := n -> \frac{2}{n^2}

x1 := proc(n) option operator, arrow; if is(n, odd) then a(n) else b(n) end if end proc
x2 := n -> a(n) * b(n)
x3 := n -> b(n) / a(n)
x4 := proc(n) option operator, arrow; if is(n, odd) then a(n) / n else b(n) end if end proc

> map(z -> evalf(z,4), [seq(x1(n), n = 1 .. 10)]);
map(z -> evalf(z,4), [seq(x2(n), n = 1 .. 10)]);
map(z -> evalf(z,4), [seq(x3(n), n = 1 .. 10)]);
map(z -> evalf(z,4), [seq(x4(n), n = 1 .. 10)]);
```
The next four lines show numerical evaluations of the four sequences that are to be summed.

[1., 0.5000, 0.3333, 0.1250, 0.2000, 0.05556, 0.1429, 0.03125, 0.1111, 0.02000]
[2., 0.2500, 0.07407, 0.03125, 0.01600, 0.009259, 0.005831, 0.003906, 0.002743, 0.002000]
[2., 1., 0.6667, 0.5000, 0.4000, 0.3333, 0.2857, 0.2500, 0.2222, 0.2000]
[1., 0.5000, 0.1111, 0.1250, 0.04000, 0.05556, 0.02041, 0.03125, 0.01235, 0.02000]

The terms in the second and third lines are terms of sequences that decrease to 0. The terms in the first and last line are not steadily decreasing.