Formulas

\[
\begin{align*}
\ln(1) &= 0, & \ln(e) &= 1, & \ln(xy) &= \ln(x) + \ln(y), & \ln(x^p) &= p \ln(x) \\
\sin\left(\frac{\pi}{6}\right) &= \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}, & \sin\left(\frac{\pi}{3}\right) &= \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}, & \sin\left(\frac{\pi}{4}\right) &= \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, \\
\sin(0) &= \sin(\pi) = 0, & \sin\left(\frac{\pi}{2}\right) &= \cos(0) = 1, & \sin\left(\frac{3\pi}{2}\right) &= \cos(\pi) = -1
\end{align*}
\]

\[
\begin{align*}
\int \frac{1}{x} \, dx &= \ln(|x|) + C, & \int \ln(x) \, dx &= x \ln(x) - x + C, & \int u \, dv &= uv - \int v \, du \\
\int \frac{1}{a^2 + x^2} \, dx &= \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C, & \int \frac{1}{\sqrt{a^2 - x^2}} \, dx &= \arcsin\left(\frac{x}{a}\right) + C, \\
\int \frac{1}{|x|\sqrt{x^2 - a^2}} \, dx &= \frac{1}{a} \text{arcsec}\left(\frac{x}{a}\right) + C
\end{align*}
\]

\[
\begin{align*}
\int \sec(x)^2 \, dx &= \tan(x) + C, & \int \csc(x)^2 \, dx &= -\cot(x) + C \\
\int \sec(x) \tan(x) \, dx &= \sec(x) + C, & \int \csc(x) \cot(x) \, dx &= -\csc(x) + C \\
\int \tan(x) \, dx &= \ln(|\sec(x)|) + C, & \int \sec(x) \, dx &= \ln(|\sec(x) + \tan(x)|) + C
\end{align*}
\]

1. Calculate (i) \( \int_{-1}^{1} \frac{-3}{x} \, dx \) (ii) \( \int_{1}^{2} \frac{9x^4 - 2x^2 + 3}{x^2} \, dx \) (iii) \( \int_{0}^{1} \sqrt{x} (x - 2)^2 \, dx \)

Solution

(i)

\[
\begin{align*}
> F := x \rightarrow \ln(\text{abs}(x)); \\
> F(x); \\
F := x \rightarrow \ln(|x|)
\end{align*}
\]
\[
\text{Answer} := F(-3) - F(-1);
\]

\[
\text{Answer} := \ln(3)
\]

(ii)

\[
\int_{1}^{2} \frac{9x^4 - 2x^2 + 3}{x^2} \, dx
\]

\[
J := \int_{1}^{2} \frac{9x^4 - 2x^2 + 3}{x^2} \, dx
\]

\[
f := \text{integrand}(J);
\]

\[
f := \frac{9x^4 - 2x^2 + 3}{x^2}
\]

\[
f := \text{expand}(f);
\]

\[
f := 9x^2 - 2 + \frac{3}{x^2}
\]

\[
F := \text{unapply}(\text{int}(f, x), x);
\]

\[
F := x \rightarrow 3x^3 - 2x - \frac{3}{x}
\]

\[
\text{Answer} := F(2) - F(1);
\]

\[
\text{Answer} := \frac{41}{2}
\]

(iii)

\[
\int_{0}^{1} \sqrt{x} (x - 2)^2 \, dx
\]

\[
J := \int_{0}^{1} \sqrt{x} (x - 2)^2 \, dx
\]

\[
f := \text{integrand}(J);
\]

\[
f := \sqrt{x} (x - 2)^2
\]

\[
f := \text{expand}(f);
\]

\[
f := x^{\frac{5}{2}} - 4x^{\frac{3}{2}} + 4\sqrt{x}
\]

\[
F := \text{unapply}(\text{int}(f, x), x);
\]

\[
F := x \rightarrow \frac{2}{7}x^{\frac{7}{2}} - \frac{8}{5}x^{\frac{5}{2}} + \frac{8}{3}x^{\frac{3}{2}}
\]

\[
\text{Answer} := F(1) - F(0);
\]

\[
\text{Answer} := \frac{142}{105}
\]
2. Calculate 
\[ F(x) = \int_{4}^{x} \frac{2^{(\frac{t}{3})}}{\sqrt{1 + \sqrt{t}}} \, dt \] 
Calculate \( F'(9) \), the derivative of \( F(x) \) at \( x = 9 \).

**Solution**

\[ F := (x) \rightarrow \text{Int}(2^{(\frac{t}{3})}/\sqrt{1+\sqrt{t}}), t = 4 \ldots x); \]
\[ F := x \rightarrow \int_{4}^{x} \frac{2^{(\frac{t}{3})}}{\sqrt{1 + \sqrt{t}}} \, dt \]
\[ D(F)(x); \]
\[ \# \]
\[ \# \text{This is calculated using the Fundamental Theorem of Calculus} \]
\[ \text{without any integration} \]
\[ \frac{2^{(\frac{\pi}{3})}}{\sqrt{1 + \sqrt{\pi}}} \]
\[ \text{derivative} := D(F)(9); \]
\[ \text{Answer} = \text{simplify( derivative )}; \]
\[ \text{derivative} := \frac{8}{\sqrt{1 + \sqrt{9}}} \]
\[ \text{Answer} = 4 \]

3. Calculate \[ \int_{0}^{\pi} \cos\left(\frac{w}{6}\right) \, dw \].

**Solution**

\[ J1 := \text{Int}(\cos(w/6), w = 0 \ldots \pi); \]
This is an "inert" (unevaluated) integral
Using "int" instead of "Int" tells Maple to leave the
integral unevaluated

\[ J1 := \int_{0}^{\pi} \cos \left( \frac{w}{6} \right) \, dw \]

> J2 := changevar( u = w/6, J1, u);
# This applies the change of variable u = w/6, du = dw/6 to
# J1
# and names the new integral J2. The integration is wrt u

\[ J2 := \int_{0}^{\pi/6} 6 \cos(u) \, du \]

> answer := value( J2 );
# This forces the evaluation of the inert integral J2

\[ answer := 3 \]

4. Calculate \[ \int_{0}^{5} (3x + 1)^{1/4} \, dx \].

Solution

> J1 := Int((3*x+1)^1/4, x = 0 .. 5);
# This is an "inert" (unevaluated) integral
# Using "int" instead of "Int" tells Maple to leave the
# integral unevaluated

\[ J1 := \int_{0}^{5} (3x + 1)^{1/4} \, dx \]

> J2 := changevar( u = 3*x+1, J1, u);
# This applies the change of variable u = 3*x+1, du = 3*dx to
J1
# and names the new integral J2. The integration is \( \frac{u}{3} \)
\[ J2 := \int_{1}^{16} \frac{u}{3} \, du \]

\[ \text{answer} := \text{value}(J2); \]
\[ \text{answer} := \text{simplify}(\text{answer}); \]
# This forces the evaluation of the inert integral J2
\[ \text{answer} := \frac{64}{15} \left( \frac{16^{1/4}}{15} - \frac{4}{15} \right) \]
\[ \text{answer} := \frac{124}{15} \]

5. Calculate
\[ \int_{e}^{e^3} \frac{1}{x \ln(x)} \, dx. \]

Solution

\[ \text{J1} := \text{Int}(1/x/\ln(x), x = \exp(1) .. \exp(3)); \]
# This is an "inert" (unevaluated) integral
# Using "int" instead of "Int" tells Maple to leave the integral unevaluated
\[ \text{J1} := \int_{e}^{e^3} \frac{1}{x \ln(x)} \, dx \]

\[ \text{J2} := \text{changevar}(u = \ln(x), \text{J1}, u); \]
# This applies the change of variable \( u = \ln(x) \), \( du = \frac{1}{x} \cdot dx \) to J1

and names the new integral $J_2$. The integration is with respect to $u$:

$$J_2 := \int_{1}^{3} \frac{1}{u} \, du$$

```plaintext
> answer := value(J2);
```

# This forces the evaluation of the inert integral $J_2$

```plaintext
answer := \ln(3)
```

6. Calculate the area of the region that is bounded above by $y = 3x$ and below by $y = x^2 + 2$.

```plaintext
> plot([3*x, x^2+2], x = 1..2, thickness = 2, color = [BLACK, NAVY]);
```

```plaintext
> Area = int(3*x - (x^2+2), x = 1..2);
```

```
Area = \frac{1}{6}
```

7. Calculate the area between the graphs of $y = x + 1$ and $y = e^x$ for $-1 \leq x \leq 1$.
8. Let \( L(x) \) denote the percentage of total earned income that is earned by the lowest \( x\% \) of all income earners. Suppose that the lowest 5\% of all income earners earn 2\% of all income and that the middle 90\% of all income earners make 50\% of all income.

Use the Trapezoidal Rule to estimate \( \int_{0}^{100} L(x) \, dx \).

**Solution**
The given information leads to the equations

\[ \begin{align*}
L(0) &= 0, & L(5) &= 2, & L(95) &= 2 + 50, & L(100) &= 100.
\end{align*} \]

The (unequal!) widths of the subintervals are 5, 90, and 5.

The Trapezoidal approximation is

\[
T = \frac{5}{2}(0 + 2) + \frac{90}{2}(2 + 52) + \frac{5}{2}(52 + 100); \quad T = 2815
\]

9. Suppose that \( g(x) \leq f(x) \) for \( 0 \leq x \leq 3 \). Use Simpson’s Rule with \( N = 4 \) and data from the following table of values to estimate the area under the graph of \( y = f(x) \) and above the graph of \( y = g(x) \) for \( 0 \leq x \leq 3 \). The given nodes are equally spaced and begin with \( x_0 = 0 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( g(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_0 )</td>
<td>5</td>
<td>4.6</td>
</tr>
<tr>
<td>( x_1 )</td>
<td>5.2</td>
<td>4.4</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>5.5</td>
<td>4.9</td>
</tr>
<tr>
<td>( x_3 )</td>
<td>5.4</td>
<td>5.2</td>
</tr>
<tr>
<td>( x_4 )</td>
<td>5.3</td>
<td>5.3</td>
</tr>
</tbody>
</table>

**Solution**

We are to approximate \( \int_{0}^{3} f(x) - g(x) \, dx \).

\[
\begin{align*}
> & \quad N := 4; \\
N := 4 \\
> & \quad \text{delta}_x := (3-0)/N; \\
\text{delta}_x := & \frac{3}{4}
\end{align*}
\]
\[
\text{Approximation} := \frac{\text{delta}\_x}{3} \left( 1\times(5-4.6) + 4\times(5.2-4.4) + 2\times(5.5-4.9) + 4\times(5.4-5.2) + 1\times(5.3-5.3) \right);
\]

\[
\text{Approximation} := 1.400000000
\]

10. Let \( F(x) = \int_{x}^{x^2} \sqrt{1+8\sqrt{t}} \, dt \). Calculate \( F'(1) \), the derivative of \( F(x) \) at \( x = 1 \).

**Solution**

Let

\[
> F := x \rightarrow \text{Int}(\text{sqrt}(1+8*\text{sqrt}(t)), t = x \ldots x^2);
\]

\[
F := x \rightarrow \int_{x}^{x^2} \sqrt{1+8\sqrt{t}} \, dt
\]

We break up the integral as

\[
\int_{x}^{36x} \sqrt{1+8\sqrt{t}} \, dt = \int_{x}^{25} \sqrt{1+8\sqrt{t}} \, dt + \int_{25}^{36x} \sqrt{1+8\sqrt{t}} \, dt
\]

There is nothing special about the number 25 in this context. Any other positive (in this problem) number would have done just as well. Today just happens to be June 25. Reversing the direction of integration of the first integral on the rhs, we obtain

\[
\int_{x}^{36x} \sqrt{1+8\sqrt{t}} \, dt = -\int_{25}^{25} \sqrt{1+8\sqrt{t}} \, dt + \int_{25}^{36x} \sqrt{1+8\sqrt{t}} \, dt
\]

Let

\[
> G := u \rightarrow \text{Int}(\text{sqrt}(1+8*\text{sqrt}(t)), t = 25 \ldots u);
\]

\[
G := u \rightarrow \int_{25}^{u} \sqrt{1+8\sqrt{t}} \, dt
\]
Then
\[ \text{eqn := } F(x) = -G(x) + G(36x) \; ]
\[ \int_{x}^{36x} \sqrt{1 + 8 \sqrt{t} \ dt} = -\int_{25}^{x} \sqrt{1 + 8 \sqrt{t} \ dt} + \int_{25}^{36x} \sqrt{1 + 8 \sqrt{t} \ dt} \]

The derivative of the first integral on the rhs of eqn is an immediate application of the Fundamental Theorem of Calculus
\[ \text{eqn2 := } \text{Diff}(G(x),x) = \text{diff}(G(x),x) \; ]
\[ \text{eqn2 := } \frac{d}{dx} \left( \int_{25}^{x} \sqrt{1 + 8 \sqrt{t} \ dt} \right) = \sqrt{1 + 8 \sqrt{x}} \]

The derivative of the second integral on the rhs of eqn is calculated by applying the Fundamental Theorem of Calculus and the Chain Rule with \( u(x) = 36x \).
\[ \text{eqn3 := } \text{Diff}(G(u(x)),x) = \text{diff}(G(u(x)),x) \; ]
\[ \text{eqn3 := } \frac{\partial}{\partial x} \left( \int_{25}^{x} \sqrt{1 + 8 \sqrt{t} \ dt} \right) = \sqrt{1 + 8 \sqrt{u(x)}} \]
\[ \text{eqn4 := } \text{Diff}(G(u(x)),x) = \text{subs}(u(x) = 36x, \text{rhs(eqn3)}) \; ]
\[ \text{eqn4 := } \frac{\partial}{\partial x} \left( \int_{25}^{x} \sqrt{1 + 8 \sqrt{t} \ dt} \right) = \left( \frac{d}{dx} (36x) \right) \sqrt{1 + 8 \sqrt{36 \sqrt{x}}} \]
\[ \text{eqn5 := } \text{lhs(eqn4)} = \text{simplify(rhs(eqn4))} \; ]
\[ \text{eqn5 := } \frac{\partial}{\partial x} \left( \int_{25}^{x} \sqrt{1 + 8 \sqrt{t} \ dt} \right) = 36 \sqrt{1 + 48 \sqrt{x}} \]

Thus
\[ \text{eqn6 := } \text{Diff}(F(x),x) = -\text{rhs(eqn2)} + \text{rhs(eqn5)} \; ]
\[ \text{eqn6 := } \frac{d}{dx} \left( \int_{x}^{36x} \sqrt{1 + 8 \sqrt{t} \ dt} \right) = -\sqrt{1 + 8 \sqrt{x}} + 36 \sqrt{1 + 48 \sqrt{x}} \]

The answer is obtained by evaluating the rhs at \( x = 1 \).
> simplify(subs(x=1, rhs(eqn6)));

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11. Calculate \( \int_{2}^{5} x^2 \sqrt{x - 1} \, dx \).

**Solution**

\[
\int_{2}^{5} x^2 \sqrt{x - 1} \, dx
\]

\[
J1 := \int_{2}^{5} x^2 \sqrt{x - 1} \, dx
\]

\[
J2 := \text{changevar}(u = x-1, J1, u);
\]

# This line says, Make the substitution \( u = x-1 \) in \( J1 \) and set \( J2 \) to be the resulting integral wrt \( u \).

# Note that the new limits of integration for \( u \) are calculated

\[
J2 := \int_{1}^{4} (1 + u)^2 \sqrt{u} \, du
\]

\[
J3 := \text{map}(\text{expand}, J2);
\]

# This expands everything in \( J2 \): First the square is expanded, then each term is multiplied by \( \sqrt{u} \).

\[
J3 := \int_{1}^{4} \sqrt{u} + 2 u^{(3/2)} + u^{(5/2)} \, du
\]

The next line evaluates this integral

\[
J1 = \text{value}(J3);
\]

\[
\text{evalf}(\text{rhs(}) \, \%);\]

\[
\int_{2}^{5} x^2 \sqrt{x - 1} \, dx = \frac{6904}{105} = 65.75238095
\]
12. The following measurements of a function have been taken, but no other values of the function are known:

\[ f(2.1) = 4, \ f(2.4) = 8, \ f(2.5) = 14, \ f(2.7) = 10, \ f(2.8) = 12, \ f(2.9) = 16, \ f(3.1) = 18. \]

Divide the interval \([2.1, 3.1]\) into subintervals (more than 1), \textit{not necessarily of equal lengths}, and, for the choice of points, use the midpoint of each subinterval to approximate

\[ \int_{2.1}^{3.1} f(x) \, dx. \]

(Everything necessary has been given. Superfluous information has also been given.)

**Solution**

The first node is the left endpoint of integration: \(x_0 = 2.0\).

The following choice of nodes is the only one possible for which we have been given all the midpoint information: \(x_0 = 2.1, \ x_1 = 2.7, \ x_3 = 3.1,\)

Then the midpoints are 2.4, and 2.9.

The subinterval lengths are \(x_1 - x_0 = 0.6\) and \(x_2 - x_1 = 0.4\).

\[
\text{approx} := \int_{2.1}^{3.1} f(x) \, dx = \left( f(2.4) \times 0.6 + f(2.9) \times 0.4 \right) \text{ approximately}
\]

or

\[
\text{approx} := \int_{2.1}^{3.1} f(x) \, dx = (0.6 f(2.4) + 0.4 f(2.9)) \text{ approximately}
\]

\[
\text{approx} := \int_{2.1}^{3.1} f(x) \, dx = 11.2 \text{ approximately}
\]
13. Calculate \[ \int_{1}^{e} x^2 \ln(x) \, dx. \]

\[
\begin{align*}
\text{> } & \quad \text{int(x^2*ln(x),x=1..exp(1));} \\
& \quad \frac{2}{9} e^3 + \frac{1}{9} \\
\text{> } & \quad \text{J1 := Int(x*ln(x),x = 1/2 .. exp(1));} \\
& \quad \# \quad \text{Int is Maple's "inert" integral. It tells Maple not to evaluate the integral. By contrast, int results in an immediate evaluation} \\
& \quad \# \quad \text{Notice that exp(1) is used in Maple for Euler's constant } e \quad \# \quad \text{(The letter e in Maple is unassigned – it has no particular significance)} \\
& \quad J1 := \int_{1/2}^{e} x \ln(x) \, dx \\
\text{> } & \quad \text{J2 := intparts(J1, ln(x));} \\
& \quad \# \quad \text{This applies integration by parts to J1 with } u = \ln(x) \text{ and } dv = x \, dx \\
& \quad J2 := \frac{1}{2}(e^2) + \frac{1}{8} \ln(2) - \int_{1/2}^{e} \frac{x}{2} dx \\
\text{> } & \quad \text{eqn := J1 = value(J2);} \\
& \quad \# \quad \text{This forces the evaluation of the inert integral} \\
& \quad eqn := \int_{1/2}^{e} x \ln(x) \, dx = \frac{1}{4}(e^2) + \frac{1}{8} \ln(2) + \frac{1}{16}
\end{align*}
\]

14. Calculate \[ \int_{0}^{\pi} x^2 \sin(x) \, dx. \]

\[
\begin{align*}
\text{> } & \quad \text{J1 := Int(x^2*sin(x),x = 0 .. Pi);} \\
& \quad \# \quad \text{Int is Maple's "inert" integral. It tells Maple not to immediately evaluate the integral. By contrast, int results in an immediate evaluation} \\
& \quad \text{Notice that } \pi \text{ is used in Maple for } \pi \text{ (The letter } \pi \text{ in Maple is unassigned – it has no particular significance.)}
\end{align*}
\]
# evaluate the integral. By contrast, int results in an immediate evaluation

\[ J_1 := \int_0^\pi x^2 \sin(x) \, dx \]

> \( J_2 := \text{intparts}(J_1, x^2); \)

# This applies integration by parts to \( J_1 \) with \( u = x^2 \) and \( dv = \sin(x) \, dx \)

\[ J_2 := \pi^2 - \int_0^\pi -2x \cos(x) \, dx \]

> \( J_3 := \text{intparts}(J_2, x); \)

# This applies integration by parts to \( J_2 \) with \( u = x \) and \( dv = 2\cos(x) \, dx \)

\[ J_3 := \pi^2 + \int_0^\pi -2 \sin(x) \, dx \]

> \text{value}(J3);

# This forces an evaluation of the inert integral

\[ \pi^2 - 4 \]

Here is a verification using Maple's \textbf{int} command.

> \text{Int}(x^2*\sin(x), x = 0 .. Pi) = \text{int}(x^2*\sin(x), x = 0 .. Pi);

# inert (unevaluated) integral on lhs, evaluated integral on rhs

\[ \int_0^\pi x^2 \sin(x) \, dx = \pi^2 - 4 \]

15. Use the reduction formula

\[ \int x \ln(x)^n \, dx = \frac{x^2 \ln(x)^n}{2} - \frac{n}{2} \int x \ln(x)^{n-1} \, dx \]

to calculate
\[
\int_{1}^{e} x \ln(x)^2 \, dx.
\]

(Remember that, by notational convention,
\[
\ln^p(x) = \ln(x)^p = (\ln(x))^p
\]
but NOT \( \ln(x^p) \).

**Solution**

\[
\begin{align*}
J & := \int x \ln(x)^n \, dx \\
J & = \int x \ln(x)^n \, dx \\
J2 & := \int \frac{1}{2} \ln(x)^n x^2 \, dx - \frac{1}{2} \ln(x)^n \int \frac{n x}{\ln(x)} \, dx \\
\text{int}(x*\ln(x)^2, x = 1 .. \exp(1)) & = -\frac{1}{4} e^2 - \frac{1}{4} \\
\text{int}(x*\ln(x)^2, x = 1 .. \exp(1)) & = -\frac{1}{4} e^2 - \frac{1}{4}
\end{align*}
\]

To simplify matters, let's convert the given reduction formula to a definite integral form.

\[
\begin{align*}
\text{subs}(x = 4, x*\ln(x)^n) - \text{subs}(x = 2, x*\ln(x)^n) & = 4 \ln(4)^n - 2 \ln(2)^n \\
\text{Reduction} & := n \rightarrow \int \ln(x)^n \, dx = 4 \ln(4)^n - 2 \ln(2)^n - n \int \ln(x)^{n-1} \, dx \\
\text{eqn1} & := \text{Reduction}(3);
\end{align*}
\]
\[ eqn1 := \int_2^4 \ln(x)^3 \, dx = 4 \ln(4)^3 - 2 \ln(2)^3 - 3 \int_2^4 \ln(x)^2 \, dx \]

> eqn2 := Reduction(2);

\[ eqn2 := \int_2^4 \ln(x)^2 \, dx = 4 \ln(4)^2 - 2 \ln(2)^2 - 2 \int_2^4 \ln(x) \, dx \]

> eqn3 := subs( eqn2, eqn1);

# \# subs( paddle = jetPack, canoe) means: wherever paddle occurs in canoe, it is replaced with jetPack
# For example, subs( x = 2*t + 1, x^2*cos(x) ) results in (2*t + 1)^2*cos(2*t + 1)

\[ eqn3 := \int_2^4 \ln(x)^3 \, dx = 4 \ln(4)^3 - 2 \ln(2)^3 - 12 \ln(4)^2 + 6 \ln(2)^2 + 6 \int_2^4 \ln(x) \, dx \]

> eqn4 := Reduction(1);

\[ eqn4 := \int_2^4 \ln(x) \, dx = 4 \ln(4) - 2 \ln(2) - \int_2^4 1 \, dx \]

> eqn5 := subs(eqn4, eqn3);

\[ eqn5 := \int_2^4 \ln(x)^3 \, dx = 4 \ln(4)^3 - 2 \ln(2)^3 - 12 \ln(4)^2 + 6 \ln(2)^2 - 24 \ln(4) - 12 \ln(2) - 6 \int_2^4 1 \, dx \]

The value of the integral on the right side is 2:

> eqn6 := Int(1,x = 2 .. 4) = value( Int(1,x = 2 .. 4) ) ;

\[ eqn6 := \int_2^4 1 \, dx = 2 \]

Therefore,

> eqn7 := subs(eqn6 , eqn5);  

\[ eqn7 := \int_2^4 \ln(x)^3 \, dx = 30 \ln(2)^3 - 42 \ln(2)^2 + 36 \ln(2) - 12 \]