Formulas

\[ \ln(1) = 0, \quad \ln(e) = 1, \quad \ln(xy) = \ln(x) + \ln(y), \quad \ln(x^p) = p \ln(x) \]

\[ \sin\left(\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}, \quad \sin\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}, \quad \sin\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, \]

\[ \sin(0) = \sin(\pi) = \cos\left(\frac{\pi}{2}\right) = 0, \quad \sin\left(\frac{\pi}{2}\right) = \cos(0) = \tan\left(\frac{\pi}{4}\right) = 1, \quad \sin\left(\frac{3\pi}{2}\right) = \cos(\pi) = -1 \]

\[ \sin(2x) = 2\sin(x)\cos(x), \quad 2\cos(x)^2 = 1 + \cos(2x), \quad 2\sin(x)^2 = 1 - \cos(2x), \]

\[ \int \frac{1}{x} \, dx = \ln(|x|) + C, \quad \int \ln(x) \, dx = x \ln(x) - x + C, \quad \int u \, dv = u v - \int v \, du \]

\[ \int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C, \quad \int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \arcsin\left(\frac{x}{a}\right) + C, \]

\[ \int \left| \frac{1}{x} \right| \sqrt{\frac{x^2}{a^2}} \, dx = \frac{1}{a} \text{arcsec}\left(\frac{x}{a}\right) + C, \quad 1 - \sin(x)^2 = \cos(x)^2, \quad 1 + \tan(x)^2 = \sec(x)^2 \]

\[ \int \sec(x)^2 \, dx = \tan(x) + C, \quad \int \csc(x)^2 \, dx = -\cot(x) + C \]

\[ \int \sec(x) \tan(x) \, dx = \sec(x) + C, \quad \int \sec(x) \cot(x) \, dx = -\csc(x) + C \]

\[ \int \tan(x) \, dx = \ln(|\sec(x)|) + C, \quad \int \sec(x) \, dx = \ln(|\sec(x) + \tan(x)|) + C \]

\[ \int \cos(x)^n \, dx = \frac{1}{n} \sin(x) \cos(x)^{(n-1)} + \frac{n - 1}{n} \int \cos(x)^{(n-2)} \, dx \]

1. Calculate \[ \int_0^4 \sin(x)^3 \cos(x)^2 \, dx. \]

Solution
> eqn1 := J = Int(sin(x)^3*cos(x)^2, x = 0 .. Pi/4);
\[
eqn1 := J = \int_{0}^{\pi/4} \sin(x)^3 \cos(x)^2 \, dx
\]

> eqn2 := J = Int(sin(x)^2*cos(x)^2*w, x = 0 .. Pi/4);

# We will begin the substitution u = \cos(x), completing it in the next line.
# Here we write sin(x)^3 as sin(x)^2*sin(x), and, to prevent Maple from combining the two factors,
# we write sin(x)^3 as sin(x)^2*w with w = sin(x) (The use of w is a temporary measure to deal
# with a feature of the software that is usually advantageous, but which is annoying here.)

\[
eqn2 := J = \int_{0}^{\pi/4} \sin(x)^2 \cos(x)^2 \, w \, dx
\]

> eqn3 := J = subs(sin(x)^2 = 1 - \cos(x)^2, rhs(eqn2) );

# Here we replace sin(x)^2 with 1 - \cos(x)^2

\[
eqn3 := J = \int_{0}^{\pi/4} (1 - \cos(x)^2) \cos(x)^2 \, w \, dx
\]

> eqn4 := J = subs(w = sin(x), rhs(eqn3) );

# Here we replace w with sin(x)

\[
eqn4 := J = \int_{0}^{\pi/4} (1 - \cos(x)^2) \cos(x)^2 \sin(x) \, dx
\]

> eqn5 := J = changevar(u = \cos(x), rhs(eqn4), u);

# This completes the change of variables u = \cos(x), \, du = -\sin(x)\, dx

\[
eqn5 := J = \int_{\sqrt{\frac{1}{2}}}^{1} (1 - u^2) u^2 \, du
\]

> eqn6 := lhs(eqn5) = map(expand, rhs(eqn5));

# Expand the integrand of the integral on the right side of eqn5
eqn6 := J = \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} u^2 - u^4 \, du

> eqn7 := \text{lhs(eqn6)} = \text{value(rhs(eqn6))};
#
# The integration is now routine

eqn7 := J = \frac{2}{15} - \frac{7\sqrt{2}}{120}

As a one-line verification, we will use Maple's built-in integrator:

> \text{Int(sin(x)^3*cos(x)^2, x = 0 .. Pi/4) = \text{value(Int(sin(x)^3*cos(x)^2, x = 0 .. Pi/4))};}

\int_{0}^{\frac{\pi}{4}} \sin(x)^3 \cos(x)^2 \, dx = \frac{2}{15} - \frac{7\sqrt{2}}{120}

For those who prefer decimals:

> \text{Int(sin(x)^3*cos(x)^2, x = 0 .. 1/4*Pi) = \text{evalf(2/15-7/120*2^(1/2));}}

0.05083754219

2. Calculate \( \int_{0}^{2} \sin(x)^2 \cos(x)^2 \, dx \).

Solution

\[
\int \sin(x)^2 \cos(x)^2 \, dx = \int (1 - \cos(x)^2) \cos(x)^2 \, dx
\]
\[
\int \sin(x)^2 \cos(x)^2 \, dx = \int \cos(x)^2 \, dx - \int \cos(x)^4 \, dx
\]
\[
\int \sin(x)^2 \cos(x)^2 \, dx = \int \cos(x)^2 \, dx - \left( \frac{\sin(x) \cos(x)^3}{4} + \frac{3 \int \cos(x)^2 \, dx}{4} \right)
\]

\[
\int \sin(x)^2 \cos(x)^2 \, dx = \frac{1}{4} \int \cos(x)^2 \, dx - \frac{\sin(x) \cos(x)^3}{4}
\]

\[
\int \sin(x)^2 \cos(x)^2 \, dx = \frac{\sin(x) \cos(x)}{2} + \frac{1}{4} \int \cos(x)^0 \, dx - \frac{\sin(x) \cos(x)^3}{4}
\]

\[
\int \sin(x)^2 \cos(x)^2 \, dx = \frac{x}{8} + \frac{\sin(x) \cos(x)}{8} - \frac{\sin(x) \cos(x)^3}{4}
\]

\[
\int_0^{\frac{\pi}{2}} \sin(x)^2 \cos(x)^2 \, dx = \frac{\pi}{8} + \frac{\sin\left(\frac{\pi}{2}\right) \cos\left(\frac{\pi}{2}\right)}{8} - \frac{\sin\left(\frac{\pi}{2}\right) \cos\left(\frac{\pi}{2}\right)^3}{4}
\]

\[
\int_0^{\frac{\pi}{2}} \sin(x)^2 \cos(x)^2 \, dx = \frac{\pi}{16}
\]

Verification using Maple's built-in integrator:

\[
> \text{Int}(\sin(x)^2*\cos(x)^2, x = 0 .. \Pi/2) = \text{value(Int}(\sin(x)^2*\cos(x)^2, x = 0 .. \Pi/2));
\]

\[
\int_0^{\frac{\pi}{2}} \sin(x)^2 \cos(x)^2 \, dx = \frac{\pi}{16}
\]

3. Express the integral \( \int \frac{3 x^2}{\sqrt{9 - x^2}} \, dx \) in the form \( A \int_{\alpha}^{\beta} \sin(\theta)^p \cos(\theta)^q \, d\theta \).

(Your answer should identify the values of \( A, p, q, \alpha, \) and \( \beta \).)
Solution

\[ \text{eqn1} := J = \int_0^3 x^2 \sqrt{9-x^2} \, dx \]

\# We need express J as an integral with integrand \( \sin^p \cos^q \)

\[ \text{eqn2} := \text{lhs(eqn1)} = \text{changevar}(x = 3\sin(t), \text{rhs(eqn1)}, t); \]

\# Make the substitution \( x = 3\sin(t), dx = 3\cos(t) \, dt \)

\[ \text{eqn2} := J = \int_0^{\pi/2} 27 \sin(t)^2 \sqrt{9 - 9\sin(t)^2} \cos(t) \, dt \]

\[ \text{eqn3} := \text{lhs(eqn2)} = \text{subs}( (9-9\sin(t)^2)^{1/2} = 3\cos(t), \text{rhs(eqn2)} ); \]

\[ \text{eqn3} := J = \int_0^{\pi/2} 81 \sin(t)^2 \cos(t)^2 \, dt \]

So, \( A = 81, p = 2, q = 2, \alpha = 0, \text{and } \beta = \frac{\pi}{2}. \)

Verification:

\[ \text{value(rhs(eqn1))}; \]

\[ \text{value(rhs(eqn3))}; \]

\[
\frac{81 \pi}{16} \\
\frac{81 \pi}{16}
\]

4. Express the integral \( \int_0^3 \frac{x^2}{(16 + x^2)^{3/2}} \, dx \) in the form \( A \int_0^\beta \sin(\theta)^p \cos(\theta)^q \, d\theta. \)
Solution

> eqn1 := J = Int(x^2/(16+x^2)^(3/2), x=0..3);

\[ eqn1 := J = \int_{0}^{3} \frac{x^2}{(16 + x^2)^{3/2}} \, dx \]

> eqn2 := J = changevar(x=4*tan(t), rhs(eqn1), t);

# Make the change of variable \( x = 4 \tan(t) \), \( dx = 4 \sec(t)^2 \, dt \)

\[ eqn2 := J = \int_{\arctan(3/4)}^{\arctan(3/4)} \frac{64 \tan(t)^2 (1 + \tan(t)^2)}{(16 + 16 \tan(t)^2)^{3/2}} \, dt \]

> eqn3 := J = subs( {1+tan(t)^2 = sec(t)^2, 16+16*tan(t)^2 = 16*sec(t)^2}, rhs(eqn2));

\[ eqn3 := J = \int_{0}^{\arctan(3/4)} \frac{1}{4} \frac{\tan(t)^2 \sec(t)^2 \sqrt{16}}{(\sec(t)^2)^{3/2}} \, dt \]

> eqn4 := J = Int(tan(t)^2/sec(t), t=0..arctan(3/4));

\[ eqn4 := J = \int_{\arctan(3/4)}^{\arctan(3/4)} \frac{\tan(t)^2}{\sec(t)} \, dt \]

> eqn5 := J = map(z -> convert(z,sincos), rhs(eqn4));

\[ eqn5 := J = \int_{0}^{\arctan(3/4)} \frac{\sin(t)^2}{\cos(t)} \, dt \]

So, \( A = 1 \), \( p = 2 \), \( q = -1 \), \( \alpha = 0 \), and \( \beta = \arctan\left(\frac{3}{4}\right) \).

The next line uses Maple's built-in numeric integrator to verify that the last integral does equal the given integral.
### 5. Express the integral \[ \int \frac{(x^2 - 4)}{x} \, dx \] in the form \( A \int \sin(\theta)^p \cos(\theta)^q \, d\theta. \)

(Your answer should identify the values of \( A, p, \) and \( q \).)

#### Solution

```maple
> eqn1 := J = Int((x^2-4)^(3/2)/x, x);
eqn1 := J = \int \frac{(x^2 - 4)^{(3/2)}}{x} \, dx

> eqn2 := J = changevar(x=2*sec(t), rhs(eqn1), t);
eqn2 := J = \int (4 \sec(t)^2 - 4)^{(3/2)} \tan(t) \, dt

> eqn3 := J = subs(4*sec(t)^2-4 = 4*tan(t)^2, rhs(eqn2));
eqn3 := J = \int 4 \sqrt{4 \tan(t)^2}^{(3/2)} \tan(t) \, dt

> eqn4 := J = 8*Int(tan(t)^4, t);
eqn4 := J = 8 \int \tan(t)^4 \, dt

> eqn5 := J = map(z -> convert(z,sincos), rhs(eqn4));
```
eqn5 := \( J = 8 \int \frac{\sin(t)^4}{\cos(t)^4} \, dt \)

\[ A = 8, p = 4, q = -4; \]

In the next line, we ask Maple to evaluate the given indefinite integral:

\[
\text{Int}((x^2-4)^{(3/2)}/x, x) = \text{int}((x^2-4)^{(3/2)}/x, x);
\]

\[
\int \frac{(x^2-4)^{(3/2)}}{x} \, dx = \frac{(x^2-4)^{(3/2)}}{3} - 4 \sqrt{x^2-4} - 8 \arctan\left( \frac{2}{\sqrt{x^2-4}} \right)
\]

Now let us ask Maple to evaluate the transformed indefinite integral that we found:

\[
eqn7 := \text{Int}((x^2-4)^{(3/2)}/x, x) = \text{value(rhs(eqn5))};
\]

\[
eqn7 := \int \frac{(x^2-4)^{(3/2)}}{x} \, dx = \frac{8}{3} \tan(t)^3 - 8 \tan(t) + 8 \, t
\]

\[
eqn8 := \text{lhs(eqn7)} = \text{subs}\{(\tan(t) = \sqrt{x^2-4}/2), \, \text{rhs(eqn7)}\};
\]

# We resubstitute to recover the original variable \( x \)

\[
eqn8 := \int \frac{(x^2-4)^{(3/2)}}{x} \, dx = \frac{(x^2-4)^{(3/2)}}{3} - 4 \sqrt{x^2-4} + 8 \arctan\left( \frac{\sqrt{x^2-4}}{2} \right)
\]

This NOT the same: compare the arctangent terms. However, it does differ by a constant, We can check by differentiating.

The derivative of the rhs of eqn8 should equal the integrand of the lhs. In other words, the difference should be 0.

\[
\text{simplify(integrand(lhs(eqn8)) - diff(rhs(eqn8), x))};
\]

0
6. What is the partial fraction decomposition of \( \frac{2x + 7}{x^2 + 3x + 2} \)?

Solution

\[
> \text{convert( \((2*x+7)/(x^2+3*x+2)\), parfrac, x);}
\]

\[
\frac{5}{x + 1} - \frac{3}{x + 2}
\]

7. What is the partial fraction decomposition of \( \frac{x^3 + 2x^2 + 3x + 4}{x^2(x^2 + x + 1)} \)?

Solution

Maple answers this question with a one-line command:

\[
> \text{convert( \((x^3+2*x^2+3*x+4)/((x^2)*(x^2+x+1))\), parfrac, x);}
\]

\[
-\frac{1}{x} + \frac{4}{x^2} + \frac{-1+2x}{x^2+x+1}
\]

The details:

\[
> \text{eqn1 := (x^3+2*x^2+3*x+4)/(x^2)/(x^2+x+1) = A/x + B/x^2 + (C*x+E)/(x^2 + x + 1);} \\
#  \\
# The form of the partial fraction expansion  \\
# eqn1 := \frac{x^3 + 2x^2 + 3x + 4}{x^2(x^2 + x + 1)} = A + B + \frac{C x + E}{x^2 + x + 1} \\
> \text{lhs(eqn1) = normal(rhs(eqn1));} \\
#  \\
# Gets a common denominator on the right side  \\
# eqn2 := \frac{x^3 + 2x^2 + 3x + 4}{x^2(x^2 + x + 1)} = \frac{A x^3 + A x^2 + A x + B x^2 + B x + B + C + C x^2 E}{x^2(x^2 + x + 1)}
\]
eqn3 := numer(lhs(eqn2)) = numer(rhs(eqn2));
#  
# Equates numerators of equal fractions, given that they have the same denominators

eqn3 := x^3 + 2 x^2 + 3 x + 4 = A x^3 + A x^2 + B x^2 + B x + B + x^3 C + x^2 E

> eqn4 := subs(x = 0, eqn3);  
#  
# The original denominator has one real root, 0. May as well substitute it to pick off one coefficient immediately

eqn4 := 4 = B

> eqn5 := subs(B = 4, eqn3);
#  
# Now that the value of B is known, substitute it back into the main identity, eqn3

eqn5 := x^3 + 2 x^2 + 3 x + 4 = A x^3 + A x^2 + 4 x^2 + 4 x + 4 + x^3 C + x^2 E

> eqn6 := lhs(eqn5) = collect(rhs(eqn5), x);
#  
# Expand and collect terms on the right side of eqn5

eqn6 := x^3 + 2 x^2 + 3 x + 4 = (A + C) x^3 + (4 + A + E) x^2 + (4 + A) x + 4

> solve( {A+C = 1, (4+A+E) = 2, 4+A = 3}, {A,C,E} );
#  
# Equate coefficients of like powers of x on both sides of eqn6. Then solve the simultaneous equations.

{A = -1, E = -1, C = 2}

8. Calculate \[ \int_0^\infty \frac{e^x}{2(1 + e^x)} \, dx. \]

Solution

> eqn1 := Int(exp(x)/((1+exp(x))^2),x = 0 .. infinity) = Limit( Int(exp(x)/((1+exp(x))^2),x = 0 .. N) , N = infinity);
\[ \text{eqn1} := \int_{0}^{\infty} \frac{e^x}{(1 + e^x)^2} \, dx = \lim_{N \to \infty} \int_{0}^{N} \frac{e^x}{(1 + e^x)^2} \, dx \]

\[ > \text{eqn2} := \text{lhs(eqn1)} = \text{Limit( changevar(u = 1 + exp(x), Int(exp(x)/((1+exp(x))^2),x = 0 .. N), u), N = infinity);} \]

\[ \text{eqn2} := \int_{0}^{\infty} \frac{e^x}{(1 + e^x)^2} \, dx = \lim_{N \to \infty} \int_{2}^{1+e^N} \frac{1}{u^2} \, du \]

\[ > \text{eqn3} := \text{lhs(eqn1)} = \text{Limit( value(Int(1/(u^2),u = 2 .. 1+exp(N))), N = infinity);} \]

\[ \text{eqn3} := \int_{0}^{\infty} \frac{e^x}{(1 + e^x)^2} \, dx = \lim_{N \to \infty} \frac{1}{1+e^N} - \frac{1}{2} \]

\[ > \text{eqn4} := \text{lhs(eqn1)} = \text{value(rhs(eqn3))}; \]

\[ \text{eqn4} := \int_{0}^{\infty} \frac{e^x}{(1 + e^x)^2} \, dx = \frac{1}{2} \]

Verification using Maple's built-in integrator:

\[ > \text{Int(exp(x)/((1+exp(x))^2),x = 0 .. infinity)} = \text{int(exp(x)/((1+exp(x))^2),x = 0 .. infinity);} \]

\[ \int_{0}^{\infty} \frac{e^x}{(1 + e^x)^2} \, dx = \frac{1}{2} \]

9. The region in the first quadrant that lies above the curve \( y = \sin(x) \) and below the curve \( y = \cos(x) \) is rotated about the x-axis. Set up an integral (or a sum of integrals) that expresses the volume of the resulting solid of revolution.
Solution

By disks:

\[
\text{eqn1} := \text{volume} = \pi \int_0^{\pi/4} \cos(x)^2 - \sin(x)^2 \, dx
\]

\[
\text{eqn2} := \text{volume} = \frac{\pi}{2}
\]

By cylindrical shells:

\[
\text{eqn3} := \text{volume} = 2\pi \int_0^{1/\sqrt{2}} y \arcsin(y) \, dy + 2\pi \int_{1/\sqrt{2}}^1 y \arccos(y) \, dy
\]

\[
\text{eqn4} := \text{volume} = \frac{\pi}{2}
\]

10. The region that lies above the curve \( y = x^2 \) for \(-1 \leq x \leq 2\) and
below the line $y = x + 2$ for $-1 \leq x \leq 2$ is rotated about the horizontal line $y = 5$. Set up an integral (or a sum of integrals) that expresses the volume of the resulting solid of revolution.

Solution

By cylindrical shells

$$\text{eqn1} := \text{volume} = 2\pi \int_{0}^{1}(5-y)\left(\sqrt{y} - \sqrt{y}\right) \, dy + 2\pi \int_{1}^{4}(5-y)\left(\sqrt{y} - (y-2)\right) \, dy$$

By washers:

$$\text{eqn3} := \text{volume} = \pi \int_{-1}^{2}(5-x^2)^2 - (5 - (x+2))^2 \, dx$$
11. The first quadrant region that lies under the curve \( y = x^2 \cos(x) \) for \( 0 \leq x \leq \pi/2 \) is rotated about the vertical line \( x = 2 \). Set up an integral that expresses the volume of the resulting solid of revolution. (Do NOT calculate the integral.)

**Solution:**

By cylindrical shells (how else?)

> eqn1 := volume = 2*Pi*Int( (2-x)*(x^2*cos(x)), x = 0 .. Pi/2);

# This line is all that was required.

\[
eqn1 := \text{volume} = 2\pi \left( \frac{\pi^3}{8} - \frac{\pi^3}{2} - 10 + 3\pi \right)
\]

An interesting number. Let’s get a floating point evaluation.

> eqn2 := volume = evalf(rhs(eqn2));

\[
eqn3 := \text{volume} = 3.039777268
\]

We will check this answer by using washers. That will require a y-integration. As a first step, let
us find the range, \([0, M]\), of integration.

The value we seek, \(M\), is the \(y\)-coordinate of the point \((c, M)\) at which \(f(x) = x^2 \cos(x)\) has a maximum in the interval \([0, \frac{\pi}{2}]\).

That is, \(c\) is in the interval \([0, \frac{\pi}{2}]\) and \(M = f(c) = c^2 \cos(c)\) and \(f(x) = x^2 \cos(x)\) is not larger for any other \(x\) in the interval.

\[
\text{For every } y \text{ in the interval } [0, M], \text{ the equation } f(x) = y \text{ has two solutions } x_1 \text{ and } x_2.
\]

The smaller solution, \(x_1\), is in the interval \([0, c]\) and the larger solution \(x_2\) is in the interval \([c, \frac{\pi}{2}]\).

The outer radius of a washer at height \(y\) is \(R = 2 - x_1\) and the inner radius is \(r = 2 - x_2\).

The volume of the washer is \(\pi (R^2 - r^2) \, dy\). We cannot find formulas for \(R\) and \(r\), but we can calculate these values numerically. The next few lines do exactly that as a first step, and then use those values to obtain a Midpoint Rule approximation to the integral that expresses the required volume using washers.

\[
\text{\textgreater} \quad \text{approx} := 0.0;
\quad \text{N} := 500:
\quad \text{Delta} := M/N:
\quad \text{for} \ j \ \text{from} \ 1 \ \text{to} \ N \ \text{do}
\quad \quad \text{s}[j] := (j-1/2)*\text{Delta}:
\quad \quad \text{R}[j] := 2 - \text{fsolve}(\text{s}[j] = f(x), x, 0..c):
\quad \quad \text{r}[j] := 2 - \text{fsolve}(\text{s}[j] = f(x), x, c..\pi/2):
\quad \quad \text{approx} := \text{approx} + \text{Delta}*(\pi*(\text{R}[j]^2 - \text{r}[j]^2)):
\quad \text{end do}:
\quad \text{printf("\nThe Volume by washers equals \%f", \text{approx});}
\]
12. A r.v. $X$ takes values in the interval $[4, 12]$. The p.d.f. $f_X$ of $X$ is given by

$$f_X(x) = \frac{c}{x^2} \quad \text{for} \quad 4 \leq x \leq 12$$

where $c$ is a positive constant.

a) What is the value of $c$?

b) What is the probability that $X$ takes on a value between 5 and 9?

c) What is the cumulative distribution function $F_X(z)$?

d) What is the mean of $X$?

e) What is the median of $X$?

\[\text{eqn1} := \int \frac{c}{x^2} \, dx \bigg|_4^{12} = 1;\]

\[\text{eqn2} := \int \frac{c}{x^2} \, dx = 1;\]

\[\text{eqn3} := c = \text{solve(eqn2, c)};\]

\[f := x \rightarrow \frac{6}{x^2};\]

b) What is the probability that $X$ takes on a value between 5 and 9?

\[\text{eqn4} := \text{answer} = \int_5^9 \frac{6}{x^2} \, dx;\]

\[\text{eqn5} := \text{map(value, eqn4)};\]
c) What is the cumulative distribution function $F_X(z)$?

\[ F(z) = \int_{4}^{z} f(x) \, dx \]

\[ F(z) = \frac{3(z - 4)}{2z} \]

d) What is the mean of $X$?

\[ \text{mean} = \int_{4}^{12} \frac{6}{x} \, dx \]

\[ \text{mean} = 6 \ln(3) \]

\[ \text{mean} = 6.591673734 \]

e) What is the median of $X$?

\[ \text{median} = \left( \frac{3}{2} \right) \left( \text{median} - 4 \right) \]

\[ \text{median} = \frac{1}{2} \]
> median = solve(eqn9, median);

\[ \text{median} = 6 \]

Verification that the median is 6:

\[
\int_{4}^{6} \frac{6}{x^2} \, dx = \frac{1}{2}, \quad \int_{6}^{12} \frac{6}{x^2} \, dx = \frac{1}{2}
\]

13. A force of 400N is required to maintain a spring so that it is stretched 3 cm beyond its rest length. (Some work was performed getting the spring to that position, but the amount of that work does not concern us.) From that position 3 cm beyond equilibrium, by performing (an additional) 144 J of work, the spring is extended even further beyond equilibrium. What is the force needed to maintain the spring at its position following the second stretching?

Let \( k \) be the spring constant.

> eqn1 := F = k*x;

\[ \text{# Hooke's Law} \]
\[ eqn1 := F = k \cdot x \]

> eqn2 := subs( \{F = 400, x = 3/100\}, eqn1);

\[ \text{# Using units of Newtons and meters} \]
\[ eqn2 := 400 = \frac{3 \, k}{100} \]

> eqn3 := k = solve(eqn2, k);

\[ eqn3 := k = \frac{40000}{3} \]
14. A hemispherical tank of radius 12 m is vertically oriented so that at the top it is a disc of radius 12 and at the bottom it is a single point. Initially it is filled with water that in the center is 10 m deep. It is pumped until the water in the center is 5 m deep. (The pumping is done so that the water reaches the top of the tank and then runs off.) Set up an integral for the work done. Use 9810 Newtons per cubic meter for the weight density.

For each \( y \) from 0 to 10, a horizontal slice of the hemisphere \( y \) meters from the top is a disc of radius \( \sqrt{144 - y^2} \). The volume of the slice is therefore

\[
\pi (144 - y^2) \, dy \quad \text{cubic meters.}
\]

The weight of that slice is

\[
9810 \pi (144 - y^2) \, dy \quad \text{N.}
\]

The work done lifting that slice \( y \) meters is

\[
9810 \pi y (144 - y^2) \, dy \quad \text{J.}
\]

The first slice that is pumped corresponds to \( y = 2 \) and the last slice that is pumped corresponds to \( y = 7 \). The total work done is therefore
\[ 9810 \pi \int_2^7 y (144 - y^2) \, dy \text{ J.} \]

This integral can be routinely evaluated (not that the calculation was requested). The amount of work done in the pumping is 81,477,794.5 Joules.

15. The horizontal sections of a tank are rectangles of width 10m.

The vertical crosssections of the tank have the equation

\[ y = 8 \left(1 - \cos(x) \right) \text{ for } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}. \]

Initially it is filled with water that in the center is 7 m deep. It is pumped until the water in the center is 2 m deep. (The pumping is done so that the water reaches the top of the tank and then runs off.) Set up an integral for the work done. Use 9810 Newtons per cubic meter for the weight density.

Solution

Note: For \( x = -\frac{\pi}{2} \) and \( x = \frac{\pi}{2} \), the value of \( y = 8 \left(1 - \cos(x) \right) \) is 8.

The height of the tank is therefore 8 meters.

Align the coordinate axes in the usual configuration with the positive x-axis pointing to the right and the positive y-axis pointing upward. The bottom of the tank is at the origin and the top of the tank is a subinterval of the horizontal line \( y = 8 \). A "slice" of water of thickness \( dy \) at level \( y \) is a rectangular box with side lengths 10, \( dy \), and 2x, where \( x \) is the abscissa of the point \((x, y)\) on the graph of \( y = 8 \left(1 - \cos(x) \right) \). Solving for \( x \) in terms of \( y \), we obtain \( x = \arccos \left(1 - \frac{y}{8}\right) \). Thus, the volume of the slice at level \( y \) is \( 2 \arccos \left(1 - \frac{y}{8}\right) 10 \, dy \) cubic meters. Its weight is
The work done pumping the slice 8 – y meters to the top of the tank is

\[ 2 \cdot (9810) \cdot (8 - y) \cdot \arccos \left(1 - \frac{y}{8}\right) \cdot 10 \, dy \, \text{J}. \]

The total work done is

\[ \int_{2}^{7} 2 \cdot (9810) \cdot (8 - y) \cdot \arccos \left(1 - \frac{y}{8}\right) \cdot 10 \, dy \, \text{J}. \]

This integral can be calculated exactly (although that calculation was not called for).

\[ \text{Int}(2 \cdot 9810 \cdot (8 - y) \cdot \arccos(1 - y/8) \cdot 10, y = 2 \ldots 7) = \int_{2}^{7} 2 \cdot 9810 \cdot (8 - y) \cdot \arccos \left(1 - \frac{y}{8}\right) \cdot 10 \, dy \, \text{J}. \]

\[ 196200 \cdot (8 - y) \left(\pi - \arccos \left(-1 + \frac{y}{8}\right)\right) \, dy = 1716750 \pi - 3041100 \arcsin \left(\frac{1}{8}\right) - 441450 \sqrt{7} - 392400 \arcsin \left(\frac{3}{4}\right) \]

or about 3,511,448.2 Joules.