NAME: Key

Math 309 Final Exam

December 15, 2016

You may use a scientific calculator, but no graphing calculators or other electronic devices allowed. You may use one hand-written (by you) 4×6 index card of notes. Fill out your scantron cards with your name (including your official "preferred" name) and your ID number. Note that each question, whether true/false or multiple choice, is worth 4 points.

Part I: True or False Determine if each of the following statements is true or false.

1. Every matrix is row equivalent to one and only one reduced echelon matrix.

   A) True B) False

2. A vector **b** is a linear combination of the columns of a matrix A if and only if the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution.

   A) True B) False

3. A set of vectors is linearly dependent if and only if one vector is a multiple of another.

   A) True B) False

4. The dimensions of the column space and the row space for an $m \times n$ matrix are equal.

   A) True B) False

5. If A is a 3×3 matrix and the eigenspaces are the x-axis, y-axis, and z-axis, then A is a diagonal matrix.

   A) True B) False

$A = PDP^{-1}$

$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ OR $PAP^{-1} = D$ $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

6. If U is an $n \times n$ (square) matrix with *orthonormal* columns, then $U^T = U^{-1}$.

A) True B) False

7. If U is an $m \times n$ matrix with *orthogonal* columns, and $\mathbf{x} \in \mathbb{R}^{\mathbf{n}}$, then $\|U\mathbf{x}\| = \|\mathbf{x}\|$.

A) True B) False

8. The sum of two eigenvectors of A is also an eigenvector of A.

A) True B) False

9. If A is a 3×3 matrix with real entries, then it must have at least one real eigenvalue.

A) True B) False

10. Every diagonalizable matrix is invertible.

A) True B) False

11. If $u,\ v \in W$, where W is a subspace of $\mathbb{R}^n$, then $proj_{\mathbf{v}}\mathbf{u} = \mathbf{u}$.

A) True B) False

12. Suppose that A is a 6×6 matrix with characteristic equation $(\lambda - 3)^3(\lambda - 1)^2(\lambda + 5)$. Then the sum of the dimensions of the eigenspaces is 3+2+1=6 and it is diagonalizable.

A) True B) False

Part II: Multiple Choice Select the best answer for each question.

13. If A and B are $n \times n$ matrices, $\det(A) = 2$ and $\det(B) = 3$, what is $\det((5A)B^T)$?

    A) 1 B) 30 C) $\frac{10}{3}$ D) $\frac{2(5^n)}{3}$ E) $6(5^n)$ F) $\frac{6}{5}$
    G) 150 H) none of the above

14. Which of the following conditions is *not* equivalent to an $n \times n$ matrix A being invertible?

    A) The number 0 is not an eigenvalue of A

    B) $\det(A) \neq 0$

    C) ColA=$\mathbb{R}^n$

    D) rank$A$=n

    E) dim Nul A=0

    F) A^T is an invertible matrix

    G) A has at least one free variable

    H) The columns of A span $\mathbb{R}^n$

    I) A is row equivalent to the identity matrix

    J) The columns of A are linearly independent

    K) none of the above

15. Consider the system of equations

$$5x_1 - 7x_2 = 12$$

$$25x_1 + hx_2 = k.$$

Give conditions on h and k such that the system has no solution.

A) $h = -35$ B) $h = -35$ and $k = 60$ C)$h \neq -35$ and $k = 60$
D) $h = -35$ and $k \neq 60$ E) $h \neq -35$ and $k \neq 60$ F) None of the above.

16. If the null space of a 5×6 matrix is four dimensional, what is the dimension of the column space?

    A) 0 B) 1 C) 2 D) 3 E) 4 F) 5 G) 6
    H) none of the above

17. Let $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ (in coordinates of the standard basis for $\mathbb{R}^3$) and let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ be an alternate basis. Find $[\mathbf{x}]_{\mathcal{B}}$.

A) $\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ B) $\begin{bmatrix} 2 \\ \frac{1}{2} \\ 1 \end{bmatrix}$ C) $\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ D) $\begin{bmatrix} \frac{5}{6} \\ \frac{1}{3} \\ 2 \end{bmatrix}$

E) none of the above

18. For a discrete dynamical system given by $\mathbf{x}_{k+1} = A\mathbf{x}_k$, the origin is a repeller. That is, the trajectories all travel outward from the origin (without spiraling). Which of the following might be the eigenvalues of the 2×2 matrix A?

A) 1.01, -0.51 B) 0.54, -2.4 C) 0.52, .75 D) 1.44, 1.2
E) 0.9+0.2i, 0.9-0.2i F) 0.8+0.7i, 0.8-0.7i G) none ~~of the above~~

19. Find a least squares solution to the inconsistent equation $Ax = b$ where

$A = \begin{bmatrix} 1 & 5 \\ 2 & -2 \\ -1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$

A) $\begin{bmatrix} 17 \\ -12 \end{bmatrix}$ B) $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \end{bmatrix}$ C) $\begin{bmatrix} \frac{1}{3} \\ \frac{8}{15} \end{bmatrix}$ D) $\begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \end{bmatrix}$ E) $\begin{bmatrix} \frac{2}{3} \\ \frac{11}{13} \end{bmatrix}$

F) $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{4} \end{bmatrix}$ G) none of the above

20. Let $V = P_2$, the vector space of degree two polynomials. Define an inner product on V by

$$f \cdot g = \int_0^1 f(x)g(x)dx.$$

In this inner product space are $p(x) = x - \frac{1}{2}$ and $q(x) = x^2 - x$ orthogonal?

A) Yes B) ~~No~~

$\int_0^1 x^3 - \frac{3}{2}x^2 + \frac{1}{2}x\,dx = \left[\frac{1}{4}x^4 - \frac{1}{2}x^3 + \frac{1}{4}x^2\right]_0^1 = -\frac{1}{2} + \frac{1}{2} = 0$

21. If $T : P_1 \to P_1$ is a transformation on linear functions such that $T(1+x) = 3+2x$ and $T(2-3x) = 11-x$, find $T(3+2x)$.

A) 75-30x B) 10+5x C) 12-3x D) 12+42x E) 12x F) None of the above

$T(1) + T(x) = 3 + 2x$

$2T(1) + (-3)T(x) = 11 - x$

22. Let $V = P_2$, the vector space of degree two polynomials. Define an inner product on V for $f(x) = ax^2 + bx + c$ and $g(x) = rx^2 + sx + t$ by

$$f \cdot g = ar + bs + ct.$$

In this inner product space are $p(x) = x^2 - 2x + 1$ and $q(x) = x^2 + 2x + 3$ orthogonal?

A) Yes B) No

$p \cdot q = 1 + (-4) + 3 = 0$

23. Let $V = C^\infty(\mathbb{R})$, the vector space of smooth functions. By the Constant Multiplier and Sum rules for derivative, we know that the second derivative D^2 is a linear transformation $D^2 : V \to V$. Which of the following functions is **not** an eigenvector for D^2?

A) $f(x) = \cos x$ B) $f(x) = \sin 2x$ C) $f(x) = 3e^x$
D) $f(x) = 3e^{2x}$ E) $f(x) = \tan x$
G) none of the above

24. Which of the following vectors is an eigenvalue of $A = \begin{bmatrix} 1 & 3 & 4 & 5 \\ 0 & -2 & 6 & 7 \\ 2 & 6 & 8 & 10 \\ 9 & -11 & -2 & 3 \end{bmatrix}$?

A) -2 B) -1 C) 0 D) 1 E) 2

F) All of the above are eigenvalues

G)More than one of the above is an eigenvalue

H) None of the above are eigenvalues

25. The vectors $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$ are linearly independent and span a two dimensional subspace W of $\mathbb{R}^3$. Use the Gram-Schmidt process to find an orthogonal basis of W including $\mathbf{v}$. What is the other vector in the basis?

A) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ B) $\begin{bmatrix} -\frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{bmatrix}$ C) $\begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ D) $\begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix}$ E) $\begin{bmatrix} 5 \\ 4 \\ 2 \end{bmatrix}$