Exam 1

Math 309 Fall 2016

October 5, 2016

No notes, calculators, or other electronic devices allowed.

Part I: Multiple Choice Each problem in this section is worth five points.

1. Let C=AB where $A = \begin{bmatrix} 10 & 2 & 17 \\ 19 & 30 & 34 \\ 12 & 36 & 36 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & \frac{3}{4} & 1 \\ \frac{5}{6} & 2 & 4 \\ 1 & 3 & 0 \end{bmatrix}$. Find c_{23}.

   A) 139 B) 189 C) 0 D) -12 E) $\frac{10}{7}$ F) None of the above.

2. Let $A = \begin{bmatrix} 17 & 31 \\ -12 & 22 \end{bmatrix}$ and $B = \begin{bmatrix} 122 & \frac{31}{89} & 12 \\ 178 & 30 & 0 \end{bmatrix}$. Among the options below, find the one for which all of the listed expressions can be legally calculated applying the rules of matrix algebra to A and B.

   A) AB, BA, and $A + B$
   B) AB and BA
   C) $A + B$ and $A - B$
   D) AB and $A + AB$
   E) AB and $B + AB$
   F) None of the above.
   G) More than one of the above

3. Consider the system of equations

$$5x_1 - 7x_2 = 12$$

$$25x_1 + hx_2 = k.$$

   Give conditions on h and k such that the system has *many* solutions.

   A) $h = -35$ B) $h = -35$ and $k = 60$ C)$h \neq -35$ and $k = 60$
   D)$h = -35$ and $k \neq 60$ E) $h \neq -35$ and $k \neq 60$ F) None of the above.

4. Which statement below is true?

   (a) Geometrically, the span of any two vectors may be seen as a plane.
   (b) The solution set of a linear system whose augmented matrix is $[\mathbf{a_1}\ \mathbf{a_2}\ \mathbf{a_3}\ \mathbf{b}]$ is the same as the solution set to the vector equation $x_1\mathbf{a_1} + x_2\mathbf{a_2} + x_3\mathbf{a_3} = \mathbf{b}$.
   (c) If a set of vectors are linearly dependent, then every vector in the set can be written as a linear combination of the other vectors in the set.
   (d) If a set of vectors are linearly dependent, then some vector in the set is a linear combination of another vector in the set.

   A) a   B) b   C) d   D) c   E) e   F) None of the above

5. If $T : P_1 \to P_1$ is a transformation on linear functions such that $T(1+x) = 3+2x$ and $T(2-3x) = 11-x$, find $T(3+2x)$.

   A) 75-30x   B) 10+5x   C) 12-3x   D) 12+42x   E) 12x   F) None of the above

6. Suppose A is an $n \times n$ matrix. Which of the following statements are NOT equivalent to A being invertible?

   (a) If T is a linear transformation and A is the standard matrix for T, then T is invertible.
   (b) The solution to the system $Ax = 0$ has *no* non-trivial solutions.
   (c) The columns of A are linearly dependent.
   (d) A has n pivots.
   (e) The transformation $T(x) = Ax$ is a one-to-one map from $\mathbb{R}^n$ to $\mathbb{R}^n$.

   A) a   B) b   C) d   D) c   E) e   F) None of the above–all are equivalent.

7. Which of the following transformations are linear?

   (a) $T(x_1, x_2) = (\sin(x_1), \cos(x_2))$
   (b) $T(x_1, x_2, x_3) = (12x_1 + 6x_2, 10x_2 + 11x_3, x_1 + x_2 + 9x_3)$
   (c) $T(x_1, x_2, x_3) = (x_1^3, x_2^2, x_3)$
   (d) $T(x_1, x_2, x_3) = (x_1, x_2, \sqrt{x_3})$
   (e) $T(x_1, x_2, x_3) = (0, 1, 2)$

   A) a B) b C) c D) d E) e F) None of the above G) More than one of the above

8. Given a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, with associated matrix A, in which of the following situations is it *possible* that T might be one-to-one?

   (a) The columns of A are linearly dependent.
   (b) $m = n$, and A^T is not an invertible matrix.
   (c) $m = n$, and the columns of A do not span $\mathbb{R}^n$.
   (d) $n > m$
   (e) $n < m$

   A) a B) b C) c D) d E) e F) None of the above G) More than one of the above

9. For a certain linear transformation T with associated matrix A, all of the following statements are true except for one. Assuming only one statement is false, which one is the false statement?

   (a) The transformation is one-to-one.
   (b) The equation $T(x) = 0$ has only the trivial solution.
   (c) $T : \mathbb{R}^n \to \mathbb{R}^m$, where $n < m$.
   (d) The columns of A are linearly dependent

   A) a B) b C) c D) d E) Based on the information, they might all be true

10. Which statement about matrices below is not true?

   (a) $(A^T)^T = A$
   (b) $(A^{-1})^{-1} = A$
   (c) $(A+B)^T = A^T + B^T$
   (d) $(A+B)^{-1} = A^{-1} + B^{-1}$
   (e) $(AB)^T = B^T A^T$
   (f) $(AB)^{-1} = B^{-1}A^{-1}$

   A) a B) b C) c D) d E) All are true F) More than one is false

11. Determine the number of free variables for the system

$$2x_1 + x_2 - 3x_3 + 13x_4 = 6$$

$$x_1 + 2x_2 + 3x_3 + 5x_4 = 12.$$

   A) 0 B) 1 C) 2 D) 3 E) 4 F) None of the above

12. Consider a transformation $T : \mathbb{R}^r \to \mathbb{R}^s$, with associated matrix A and B the row-echelon reduction of A. If T is onto, which of the following might be true?

   (a) $r = 7$, $s = 5$ and B has five pivots.
   (b) $r = 5$, $s = 7$ and B has five pivots.
   (c) $r = 5$, $s = 7$ and B has four pivots.
   (d) $r = 7$, $s = 5$ and B has four pivots.
   (e) T is not onto.

   A) a B) b C) c D) d E) e F) None of the above

Part II: Hand graded

Show all appropriate work to receive full credit. Each question is worth 10 points.

1. A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ first rotates points around the origin by an angle of $\frac{\pi}{2}$ radians, then reflects points through the line $x_2 = x_1$.

   (a) Give the matrix A_1 for the transformation which rotates points around the origin by an angle of $\frac{\pi}{2}$ radians.

   (b) Give the matrix A_2 for the transformation which reflects points through the line $x_2 = x_1$.

   (c) Give the matrix A_T for the T described above.

   (d) Describe the relation between A_1, A_2, and A_T, showing any appropriate calculations supporting your claim.

2. Clearly state whether each statement is true or false. If the statement is true, give a proof. If the statement is false, give a counterexample or a reason why it cannot be true.

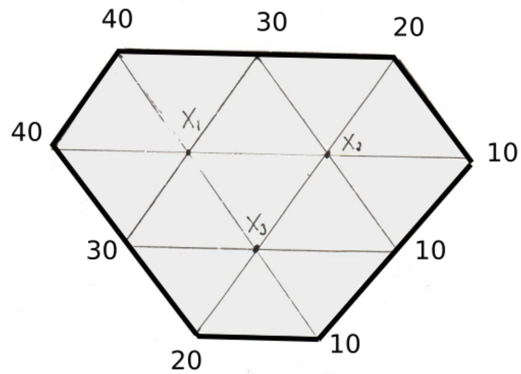
   (a) If A and B are square $(n \times n)$ matrices, then

   $$AB = BA.$$

   (b) If A and B are invertible $n \times n$ matrices, then

   $$(AB)^{-1} = B^{-1}A^{-1}.$$

   (c) If A is invertible, then $Ax = b$ has the unique solution $x = A^{-1}b$.



3. In the figure above temperatures are given at edge points for a non-regular hexagonal plate. (You may assume all of the interior triangles are equilateral.) Use a variation on the heat transfer model used in class to answer the following questions.

   (a) Set up a system of linear equations in **three** variables which gives the temperature at points x_1, x_2, and x_3.

   (b) Give the augmented matrix for your system.

   (c) Solve for x_1, x_2, and x_3, either by row reducing the system or by a shorter calculation taking advantage of any symmetries you might observe.

4. For each matrix below, either give a reason why it is not invertible or find the inverse.

(a) $A = \begin{bmatrix} 4 & 3 \\ 5 & 4 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & -3 & 17 \\ 4 & 20 & 6 \\ 3 & -9 & 51 \end{bmatrix}$

(c) $B = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$